FLUCTUATION-FLIPPING ORBITS
OF FREELY-DRAINING RIGID DUMBBELLS
IN CONVERGING-DIVERGING PORE FLOWS*

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This paper addresses the deterministic translational-rotational motion of a freely-draining rigid dumbbell that is freely suspended in pressure-driven Stokes flow through a periodically corrugated pore or channel. As is demonstrated by numerical integration of the coupled, nonlinear equations of motion within a sinusoidally corrugated, axisymmetric pore, passage of the dumbbell through successive converging and diverging local flow environments engenders rotary fluctuations relative to the local streamlines that gradually increase in amplitude, over many pore wavelengths, until it eventually flips over. Then the dumbbell slowly realigns itself with the flow, with diminishing fluctuations—whereupon the whole cycle begins again.

To analyze the cumulative effect of zero-mean rotary fluctuations, we first consider a simpler problem in which a prolate body of zero axis ratio is subjected to externally applied, harmonic rotary oscillations while immersed in steady, homogeneous shear. Two-timescale asymptotics combined with inner-outer matching yield a leading-order formula for the orientational history that agrees very closely with the numerical solution for this case. In particular, the non-oscillatory (slow) component of the behavior is asymptotically equivalent to a Jeffery orbit described by a nonzero effective axis ratio.

Subsequently the hydrodynamic mechanism responsible for moving the dumbbell (of dimensionless length $2\alpha$) out of alignment with the local streamlines is analyzed for a two-dimensional, periodically corrugated channel, with accounting for (i) the leading effect of spatial inhomogeneity of the flow field, and (ii) the first two corrections in (small) angle of misalignment. For sufficiently short times following the initial condition of perfect alignment with the flow, the trajectory of the dumbbell’s midpoint is effectively decoupled from the orientational history. These approximations lead to a Riccati equation, for which the coefficients are periodic functions under a subtle assumption involving the flow field. A two-scale analysis then explains the numerically observed $t^{-1/3}$ scaling of the distance traveled between successive flips. Motivation for the required assumption, and approximations of the hydrodynamic coefficients, are obtained from lubrication theory. When a sinusoidally corrugated, two-dimensional channel is chosen to match the meridian-plane generator of the axisymmetric pore, the asymptotic orientational dynamics for the former and the numerical solution for the latter are found to appear remarkably similar.


KEYWORDS Freely-draining rigid dumbbell Polymers Orientational dynamics Jeffery orbits Method of multiple scales Matched asymptotic expansions Lubrication theory Stokes flow Riccati equation with periodic coefficients

*Respectfully dedicated to Professor Howard Brenner in celebration of his 65th birthday.
1. INTRODUCTION

This paper addresses the orientational dynamics of a freely-draining rigid dumbbell that is carried along by pressure-driven Stokes flow through a periodically corrugated axisymmetric tube or two dimensional channel. Our analysis is motivated by a previous numerical investigation (Nitsche, 1996), in which an elastic, freely-draining dumbbell in a sinusoidally corrugated tube was observed to undergo an unexpected fluctuation-flipping behavior. Even if the dumbbell started out perfectly aligned with the local velocity vector, its passage through successive throats and bodies engendered small-amplitude rotary fluctuations relative to the local streamlines. These oscillations increased in amplitude until, after traveling many wavelengths, the dumbbell eventually flipped over, and then returned slowly (with diminishing fluctuations) toward the aligned condition—whereupon the whole cycle began again.

With reference to the dynamics of polymer solutions, we mention that misalignment of an elastic dumbbell relative to the local streamlines of an inhomogeneous flow field—whether it arises from deterministic advection or from Brownian motion—is known to be crucial for cross-stream migration (Shafer et al., 1974; Aubert and Tirrell, 1980; Sekhon et al., 1982; Brunn, 1983; Nitsche, 1996). This mechanism plays an important role in the literature on polymer migration, which is reviewed by Jhon et al. (1987), Larson (1992) and Agarwal et al. (1994).

In the context of polymer migration, only Jeffery-orbit tumbling due to shear seems previously to have been advanced (Brunn, 1983) as a deterministic mechanism for the crucial flow-dumbbell misalignment, which was subsequently incorporated into the kinetic theory (Brunn and Chi, 1984). The required nonzero axis ratio entered through third reflections between the beads. Unfortunately, the ubiquitous freely-draining dumbbell model of polymers (Bird et al., 1987) does not tumble in the two kinds of inhomogeneous flow fields most commonly used in theories of cross-stream migration: rectilinear tube or channel flow, and rotary Couette flow between concentric cylinders. Rotary fluctuations due to Brownian motion are then a necessary ingredient in lateral migration.

The fluctuation-flipping behavior originally observed numerically in the corrugated tube is inseparably tied to the passage of the dumbbell between different local flow environments. It arises partly because the process of rotary relaxation can never quite "outrun" translational advection, and does not hinge on elasticity— as will be seen from analogous numerical computations in §2 for a rigid dumbbell. Here we shall develop an asymptotic theory that explains the underlying physical mechanism for the rigid case. Compared with the dynamics and kinetic theory of flexible dumbbells and other entities, rigid particles in inhomogeneous flows seem to have received much less attention (to be discussed in §2).

In §3 we consider a spatially homogeneous problem of externally-forced, small-amplitude rotary oscillations superposed upon the angular velocity due to steady shear, for which a two-timescale analysis combined with matched asymptotic expansions yields a uniformly valid approximation at leading order. On the slow timescale (nonoscillatory component of the asymptotic solution), the freely-draining dumbbell behaves as if it were endowed with a nonzero effective axis ratio; the period of the resultant Jeffery-like orbits scales inversely with the amplitude of the oscillatory
forcing. Within the general framework of dynamical systems advanced by Szeri et al. (1991, 1992, 1993) for suspended microstructure in "complex" flows, our simple example in §3 corresponds to a quasiperiodic attractor of a nonautonomous ODE, for which "the 'phase' on [this] ... attractor depends on initial orientation" (Szeri and Leal, 1993, p. 166). Szeri et al. (1991, pp. 221–223) illustrate the principle that a slow time-dependent term incorporated into an otherwise autonomous differential equation can lead to global changes in qualitative behavior. Here we shall observe an analogous effect of small-amplitude (periodic) forcing; cf. Szeri et al. (1992, pp. 43–46).

Hydrodynamic origins of the rotary fluctuations of the dumbbell relative to the converging-diverging streamlines are considered in §4 for a two-dimensional, periodically corrugated channel. Using truncated expansions in small angle of misalignment and short length of the dumbbell, we ultimately obtain a simple nonlinear ODE (Riccati equation with fluctuating coefficients) that captures the essential mechanistic features of the time-varying flow environment sampled by the dumbbell. Under a subtle assumption involving the flow field, by which the coefficient functions become periodic, a two-scale analysis of the incipient stages of misalignment (§5)—analogous to the outer solution in §3—explains why the distance traveled between flips should scale inversely with the length of the dumbbell, as was observed numerically. The relevant hydrodynamic coefficients are approximated using lubrication theory in §6. A resulting illustrative plot of the asymptotic orientational history is strikingly similar to the numerical results from §2 (cf. Nitsche, 1996)—despite the obvious difference between the two-dimensional vs axisymmetric geometries (which match in a meridian plane of the latter).

The work of Szeri et al. (1991, 1992, 1993) emphasized the inherently nonautonomous nature of Lagrangian unsteadiness; but the role of spatial inhomogeneity of the flow was only to provide a smooth variation of local environments to be experienced by the suspended microstructure in being carried from one place to another. In particular, any direct effect of spatial inhomogeneity was neglected in their dynamic equations, which involved only the local gradient of the velocity field but no higher corrections for appreciable size of the suspended microstructure compared with the characteristic lengthscale of the inhomogeneous flow. This assumption would seem initially to be unassailable for sufficiently small suspended particles. However, we shall see that the crucial term responsible for driving the dumbbell out of alignment with the flow is, in fact, proportional to the square of the ratio of microscopic to macroscopic lengthscales. In other words, spatial inhomogeneity of a steady flow field cannot, in general, be separated from its manifestation in Lagrangian unsteadiness: it is not a lower-order effect (with regard to global behavior).

2. FLOW IN A SINUSOIDALLY CORRUGATED PORE

A classic series of papers developed the behavior of rigid axisymmetric particles in homogeneous shear (Jeffery, 1922; Bretherton, 1962; Brenner, 1964a), for which the salient aspects of particle shape are encapsulated in a single parameter—the equivalent axis ratio of a spheroid [see also Brenner (1972)]. Foundations for the motion of an arbitrarily-shaped rigid particle in a general inhomogeneous flow field were given by
Professor Brenner (1964ii). Some exact analytical solutions for spheroids in quadratic flows were derived by Chwang (1975) using a singularity method. More recently, detailed calculations of the translational-rotational motion of rigid rods in various inhomogeneous flows have been carried out with slender body theory by Shanker et al. (1991) and Pittman and Kasiri (1992). Most relevant to our problem were their results on (i) actual rotation rate compared with that obtained from the local, linearized flow field at the center or an approximation based upon an effective local shear rate; and (ii) motion relative to the local fluid velocity at the center. Nitsche (1993) has considered more complex, cellular flow fields for rigid, freely-draining dumbbells (i.e., having point-sized beads) that in some cases straddle the cellular scale. Results from his invariant formulation are used here in a simpler, two-dimensional Cartesian representation to describe the motion of the dumbbell (of dimensionless length 2) in a meridian plane of the sinusoidally corrugated tube depicted in Figure 1. Thus, we write the linear and angular velocity of the dumbbell as follows in terms of the (undisturbed) fluid velocity vectors at both ends:

\[
\dot{x} = \frac{1}{2}(u^{+1} + u^{-1}), \quad (1)
\]

\[
\dot{y} = \frac{1}{2}(v^{+1} + v^{-1}), \quad (2)
\]

\[
\dot{\phi} = \frac{1}{2}\{\cos \phi (u^{+1} - u^{-1}) - \sin \phi (v^{+1} - v^{-1})\}, \quad (3)
\]

with

\[
u^{±1}(x, y, \phi) = u(x ± \ell \cos \phi, y ± \ell \sin \phi) \quad (4)
\]

**FIGURE 1** Dimensionless unit cell of the axisymmetric, sinusoidally corrugated pore model, for which the Stokes equations are solved with a least-squares boundary singularity method, slightly modified from Nitsche and Brenner (1990). Open circles represent the locations of ring singularities that form the basis of the solution by linear superposition. Solid circles indicate the points in the meridian plane at which the no-slip, periodic and/or pressure-drop boundary conditions are imposed for the unit cell. Referred to unit maximum radius, the wavelength and minimum radius of the wall corrugations are \(\lambda = 2\) and \(R_{\text{min}} = 0.6\), respectively.
and similarly for \(v^{\pm} (X, \Psi, \phi)\). Because of the point-sized beads, hydrodynamic wall effects (Happel and Brenner, 1983, chap. 7) are negligible even when the dumbbell is not short compared with the pore scale.

The (steady) axisymmetric Stokes flow field in the sinusoidally corrugated tube—the same as that considered for flexible dumbbells by Nitsche (1996)—is calculated using a slight modification of the least-squares boundary singularity scheme described by Nitsche and Brenner (1990). By normalization we have unit axial velocity averaged over the unit cell. A fourth-order Runge–Kutta scheme (step size \(\delta t = 0.001\)) is then applied to integrate the coupled equations (1)–(3) numerically.

With reference to Figure 2, we denote by \(\alpha\) the angle between the dumbbell axis and the fluid velocity vector at its midpoint. Figure 3 shows \(\alpha\) as a function of axial distance \(X\) traveled by the midpoint. The distance between successive end-for-end flips (over which many smaller fluctuations occur) is seen to scale inversely with the length of the dumbbell. The orientational behavior appears essentially the same for the rigid vs elastic dumbbells [cf. Nitsche (1996)]—provided that the duration of observation is not so long that (slow) lateral migration in the latter case moves the elastic dumbbell into streamlines of significantly higher shear, thereby decreasing the flipping distance.

3. OSCILLATORY FLUCTUATIONS AND FLIPPING BEHAVIOR

Before modeling in detail how curvilinear streamlines lead to the fluctuating misalignment of a freely-draining dumbbell relative to the local flow direction (§4, below), we first analyze the effect of rotary fluctuations in an artificial (but physically realizable) case that can immediately be described with one ODE: externally forced rotary oscillations superposed upon the angular velocity due to steady simple shear flow. The
fact that the dumbbell experienced variations in shear rate while passing between throats and bodies in the corrugated pore of §2 is of secondary importance for our idealized problem: it is the angular fluctuations that are crucial for the periodic flipping behavior we seek to explain. In the dynamic considerations of §4, such an artificial separation of effects will, however, be impossible. The period of the oscillations and the rotary timescale are both inversely proportional to the characteristic flow velocity; they must be of similar magnitude because—except in §6—the wavelength $\lambda$ of the wall corrugations is comparable to the maximum radius (unity) of the pore.

FIGURE 3 Dumbbell-streamline angle $\alpha$ (Figure 2) plotted as a function of axial distance traveled $X$ for flow through the corrugated tube from Figure 1. (a) Comparison for $\ell = 0.05$ vs $\ell = 0.025$. (b) Detailed view of flipping behavior for $\ell = 0.05$. (c) Detailed view of near-alignment behavior for $\ell = 0.05$. 
A prolate body with zero equivalent axis ratio will, if left undisturbed, simply become aligned with the surrounding shear flow. The central point here is that the zero-mean external fluctuations, acting over many cycles, slowly carry the particle across the aligned state. Indeed, the (slow) nonoscillatory component of the trajectory for small amplitude will be shown to be asymptotically equivalent to a Jeffery orbit, whereby the imposed fluctuations can be regarded as conferring upon the body a nonzero effective axis ratio.

With the freely-draining dumbbell confined (by virtue of the initial condition) to the plane of shear, its configuration can be described completely by the azimuthal angle $\phi(t)$. We add small-amplitude, harmonic angular forcing to the usual description of rotary convection in steady shear for a prolate body with zero axis ratio ratio [see, e.g., Leal and Hinch (1971) and Kim and Karrila (1991, §5.5.2)]:

$$\frac{d\phi}{dt} = \gamma \left[ \sin^2 \phi + \varepsilon \cos(t) \right] \quad (\varepsilon \ll 1),$$

with $\varepsilon$ a rotary amplitude and $\gamma$ the shear rate. For illustrative purposes we take $\gamma = 1$; note that the rotary timescale is roughly similar to the period of the harmonic forcing. The dumbbell starts off aligned with the shear flow:

$$\phi(0; \varepsilon) = 0.$$

Numerical solutions for $\phi(t; \varepsilon)$ are shown in Figure 4 for $\varepsilon = 1^\circ, 2^\circ, 4^\circ$; these are carried out with a fourth-order Runge-Kutta scheme using a suitably small time step ($\Delta t = 0.001$). One observes the actual flipping to occur on the $\text{ord}(1)$ timescale of rotary relaxation ("inner" solution). Successive flips are separated by a period scaling like $\varepsilon^{-1}$, throughout most of which $\phi(t; \varepsilon)$ fluctuates very near zero ("outer" solution). This behavior can be treated with matched asymptotic expansions—a complicating element...
being that the outer solution requires a two-time expansion for uniform validity on an \( \text{ord}(\varepsilon^{-1}) \) timescale. By this plan we now proceed.

A regular perturbation in \( \varepsilon \) starting at \( t = 0 \) gives the expansion

\[
\phi^{(0)}(t; \varepsilon) = \varepsilon \sin t + \varepsilon^2 \left( \frac{1}{2} t - \frac{1}{4} \sin 2t \right) + \varepsilon^3 \left( \frac{3}{4} \sin t + \frac{1}{12} \sin 3t - t \cos t \right) + \ldots ,
\]

for which asymptotic ordering of the terms is seen to break down at \( \text{ord}(\varepsilon^{-1}) \) times.

[Here we use the symbol "ord" to mean "strictly of order", as defined by Hinch (1991, p.6).] Thus, we should use a two-time expansion (Bender and Orszag, 1978, chap. 11; Nayfeh, 1973, chap. 6) for the outer solution

\[
\phi^{(0)}(t; \varepsilon) \sim \phi^{(0)}_0(\tau, T) + \varepsilon \phi^{(0)}_1(\tau, T) + \varepsilon^2 \phi^{(0)}_2(\tau, T) + \varepsilon^3 \phi^{(0)}_3(\tau, T) + \ldots ,
\]

with \( \tau = t \) and \( T = \varepsilon t \). The hierarchy of equations follows in a standard way from collecting terms of like powers of \( \varepsilon \):

\[
\frac{\partial \phi^{(0)}_0}{\partial \tau} = \sin^2 \phi^{(0)}_0 ,
\]

\[
\frac{\partial \phi^{(0)}_1}{\partial \tau} = - \frac{\partial \phi^{(0)}_0}{\partial T} + \cos \tau + \phi^{(0)}_1 \sin (2\phi^{(0)}_0) ,
\]

\[
\frac{\partial \phi^{(0)}_2}{\partial \tau} = - \frac{\partial \phi^{(0)}_1}{\partial T} + \phi^{(0)}_2 \sin (2\phi^{(0)}_0) + (\phi^{(0)}_1)^2 \cos (2\phi^{(0)}_0) ,
\]

\[
\frac{\partial \phi^{(0)}_3}{\partial \tau} = - \frac{\partial \phi^{(0)}_2}{\partial T} + \phi^{(0)}_3 \sin (2\phi^{(0)}_0) + 2\phi^{(0)}_1 \phi^{(0)}_2 \cos (2\phi^{(0)}_0) - \frac{2}{3} (\phi^{(0)}_1)^3 \sin (2\phi^{(0)}_0) .
\]
The initial condition (6) means we must require that
\[ \phi^{(O)}_i(0,0) = 0 \quad (i = 0, 1, 2, 3, \ldots). \]  
(13)

Regarding the slow time variable \( T \) as a parameter, any nontrivial solution of Equation (9) can be written in the following form:
\[ \phi^{(O)}_0(\tau, T) = \arccot[C(T) - \tau]. \]  
(14)

As the initial condition (13) would require \( C(0) \) to be unbounded, we find only the trivial solution at zeroth order:
\[ \phi^{(O)}_0(\tau, T) \equiv 0. \]  
(15)

At first order we then obtain
\[ \phi^{(O)}_1(\tau, T) = \sin \tau + A(T). \]  
(16)

Elimination of the secular term (linear growth in \( \tau \)) at second order, Equation (11), determines \( A(T) \):
\[ \phi^{(O)}_2(\tau, T) = \left\{ -A'(T) + \frac{1}{2} + [A(T)]^2 \right\} \tau - 2A(T) \cos \tau - \frac{1}{4} \sin 2\tau + B(T), \]  
(17)

where the initial condition \( A(0) = 0 \) yields
\[ A(T) = (1/\sqrt{2}) \tan (T/\sqrt{2}). \]  
(18)

The secular term at third order is given by
\[ \phi^{(O)}_3(\tau, T) = \{-B'(T) + 2A(T)B(T)\} \tau + \{\text{nonsecular terms}\}, \]  
(19)

whereby it follows that
\[ B(T) = B(0) \exp \left\{ 2 \int_0^T A(s) \, ds \right\} \equiv 0. \]  
(20)

Thus, we obtain the two-timescale expansion,
\[ \phi^{(O)}(t, \varepsilon) \sim \varepsilon \phi^{(O)}_1(t, \varepsilon t) + \varepsilon^2 \phi^{(O)}_2(t, \varepsilon t) + \ldots \]
\[ = \varepsilon \left[ \sin \frac{t}{\sqrt{2}} \tan \left( \frac{\varepsilon t}{\sqrt{2}} \right) \right] \]
\[ - \varepsilon^2 \left[ \frac{1}{4} \sin 2\varepsilon t + \sqrt{2} \cos \varepsilon t \tan \left( \frac{\varepsilon t}{\sqrt{2}} \right) \right] + \ldots, \]  
(21)

which is uniformly valid on the interval \( 0 \leq t \leq \varepsilon^{-1} \) for any positive \( \varepsilon < \pi/\sqrt{2} \).

This formula cannot describe the flipping of the dumbbell \((\phi = \pi/2)\) because reaching \( \text{ord}(1) \) values violates, by definition, the ordering of an expansion that begins with the \( O(\varepsilon) \) term. Both of the functions \( \phi^{(O)}_1(t, \varepsilon t) \) and \( \phi^{(O)}_2(t, \varepsilon t) \) blow up at time \( t^* = (\pi/\sqrt{2})\varepsilon^{-1} \), essentially "overshooting" in what could be regarded as an "attempt" by the outer expansion to model behavior in the inner region. It should be noted that the asymptotic ordering of the first two terms is preserved as \( t \to t^* \); cf. Nayfeh (1973, p. 231).
Unboundedness of the outer solution at \( t = t^* \) hints that the flip \( (\phi = \pi/2) \) should occur precisely at that time. This expectation remains to be verified by the subsequent inner-outer matching process; for now we shall write the inner expansion with the time \( \ell \) left unspecified in the end condition

\[
\phi^{(U)}(\ell; \varepsilon) = \frac{\pi}{2}.
\]

(22)

A regular perturbation for the inner region, valid for \( 0 \leq \ell - t = O(1) \), yields

\[
\phi^{(I)}(t; \varepsilon) \sim \phi^{(I)}_0(t) + \varepsilon \phi^{(I)}_1(t) + \ldots
\]

\[
= \text{Arc} \cot (\ell - t) + \varepsilon \left\{ \left( \frac{(\ell - t)^2 - 1}{(\ell - t)^2 + 1} \right) \sin t - \left[ \frac{2(\ell - t)}{(\ell - t)^2 + 1} \right] \cos t \right\} + \ldots
\]

(23)

For the inner behavior of the outer solution and the outer behavior of the inner solution we define the respective relative time variables \( q \) and \( s \) as follows:

\[
t = t^* - q = \ell - s.
\]

(24)

Thus we find the leading-order asymptotic behaviors,

\[
\varepsilon \phi^{(O)}_i(t; \ell t) \sim \frac{1}{q} \varepsilon \sin t + O(\varepsilon^2 q),
\]

\[
\phi^{(O)}_0(t) + \varepsilon \phi^{(O)}_1(t) \sim \frac{1}{s} \varepsilon \sin t + O(\varepsilon/s) + O(s^{-2}).
\]

(25)

These two forms are brought into correspondence with each other by choosing \( \ell = t^* \) so that \( s = q \), i.e., we should apply the end condition (22) for the inner solution precisely where outer solution blows up. Throughout an overlap region defined by

\[
\varepsilon^{-a} < q < \varepsilon^{-b} \quad (1/2 < a < b < 1),
\]

(26)

we have, modulo \( o(\varepsilon) \) discarded terms, the common asymptotic behavior

\[
\phi^{(M)}(t; \varepsilon) = \frac{1}{t^*(\varepsilon) - t} + \varepsilon \sin t,
\]

(27)

with the flipping time \( t^*(\varepsilon) = (\pi/\sqrt{2})\varepsilon^{-1} \). Thus, for the uniformly-valid approximation at leading order we find

\[
\phi^{(U)}(t; \varepsilon) = \varepsilon \phi^{(O)}_1(t; \ell t) + \left[ \phi^{(O)}_0(t) + \varepsilon \phi^{(O)}_1(t) \right] - \phi^{(M)}(t; \varepsilon)
\]

\[
= \frac{\varepsilon}{\sqrt{2}} \tan \left( \frac{\ell t}{\sqrt{2}} \right) + \text{Arc} \cot (t^* - t) - \frac{1}{t^* - 1}
\]

\[
+ \varepsilon \left\{ \left( \frac{(t^* - t)^2 - 1}{(t^* - t)^2 + 1} \right) \sin t - \left[ \frac{2(t^* - t)}{(t^* - t)^2 + 1} \right] \cos t \right\}.
\]

(28)

Figure 5 illustrates very close agreement between the numerical and asymptotic approximations for \( \varepsilon = 4^\circ \). In the nearly aligned configuration \( (\phi \approx 0) \), what eventually swings the dumbbell past the asymptote \( \phi = 0 \) of the inner solution is the fact that both positive and negative fluctuations lead to a positive derivative-contribution of magnitude \( \text{ord}(\phi^3) \). Here a vague analogy exists with Brownian fluctuations in rotary
advection–diffusion (Leal and Hinch, 1971; Hinch and Leal, 1972i, ii; Layec, 1972; Stewart and Sorensen, 1972).

The Jeffery orbit for a prolate spheroid of (minor/major) axis ratio $R$ confined to rotate in the plane of shear is described by the equation

$$\frac{d\phi}{dt} = \frac{\gamma}{1 + R^2}\left[\sin^2 \phi + R^2 \cos^2 \phi\right], \quad (29)$$

with $\gamma$ the shear rate. The solution is well known (see, e.g., Karrila and Kim, 1991, §5.5.2):

$$\phi(t; R) = \arctan\left[\frac{1}{R} \tan\left(\frac{\gamma t}{R + R^{-1}}\right)\right]. \quad (30)$$

In order to illustrate similarities with the oscillatory problem, we consider the asymptotic behavior in the limit as $R \to 0$ for the case $\gamma = 1$. Calculations analogous to those above give the following uniformly-valid approximation upon matching the inner and outer solutions:

$$\phi^{(\infty)}(t; R) = R \tan(Rt) + \text{Arccot}(t^\circ - t) - \frac{1}{t^\circ - t} \quad (31)$$

with the half period $t^\circ = (\pi/2)R^{-1}$. For aspect ratio $R = \varepsilon/\sqrt{2}$ and the same shear rate ($\gamma = 1$), the Jeffery orbit is seen to be asymptotically identical with the nonoscillatory terms in Equation (28). In other words, when a prolate body of zero axis ratio undergoes externally-forced, harmonic rotary fluctuations of amplitude $\varepsilon$ superposed upon unit steady shear, it behaves on the slow timescale as if it were endowed with the effective axis ratio $\varepsilon/\sqrt{2}$ while experiencing only the same steady shear rate.
Finally, it should be remarked that the above asymptotic scheme applies equally to any functional form \( f(t) \) of the (zero-mean) rotaty fluctuations. The only difference in the slow-scale behavior at leading order would come from replacing \( \langle \sin^2 t \rangle = 1/2 \) with the appropriate value of \( \langle [F(t)]^2 \rangle - F(t) \) being the antiderivative of \( f(t) \). The analysis can also readily be extended to an arbitrary shear rate \( \gamma \).

4. THE ORIGIN OF ROTARY FLUCTUATIONS IN THE NEARLY-ALIGNED REGIME

In the simple model of §3 we observe—see Equation (28) and Figures 4 and 5—that the angle \( \phi \) stays within \( \varepsilon \) of the aligned state \( (\phi = 0, \pi, 2\pi, \ldots) \) for an ord \( (\varepsilon^{-1}) \) interval of time, and so the dumbbell rocks back and forth across the streamwise direction many times before eventually flipping over. This feature does not realistically reflect behavior within the corrugated pore (Figure 3c), where the amplitude of the rotary oscillations decreases as the dumbbell approaches alignment and then increases again once it has swung past: thereby the function \( \alpha(x) \) crosses 180° at most a few times. To understand this more subtle behavior we must abandon the simple notion of externally imposed rotary oscillations and analyze in detail the dynamic origins of the fluctuating misalignment of the dumbbell relative to the converging-diverging streamlines that carry it through the pores. This is a case where the usual assumption of local linearity of the flow (see e.g., Harris and Pittman, 1976; Shaqfeh and Koch, 1988; Szeri et al., 1991, 1992; Szeri and Leal, 1993, 1994) cannot be used: the dimensionless half-length \( \varepsilon \) of the dumbbell emerges as an important parameter—owing to the subtle but crucial role played by one particular \( O(\varepsilon^2) \) term resulting from a local Taylor expansion of the velocity field about the dumbbell's midpoint. With suitable approximations, it will turn out that the orientational fluctuations of the dumbbell can be modeled using a Riccati equation (Ince, 1956, §2.15) with periodic coefficients.

With reference to the coupled dynamic Equations (1)–(3), we regard the midpoint of the dumbbell at any time \( t \) as the origin \( P = P(t) \) of local \( (\xi, \psi) \) Cartesian coordinates, which are oriented such that the \( \xi \) axis lies parallel to the fluid velocity vector at \( P \). In this coordinate system, the \( \xi \) and \( \psi \) components of velocity are denoted by the functions \( U(\xi, \psi) \) and \( V(\xi, \psi) \). When used as subscripts, the symbols \( \xi \) and \( \psi \) denote partial derivatives. We shall use the script letters \( \mathcal{U} \) and \( \mathcal{V} \) to indicate that the fluid velocity components or their various partial derivatives are being evaluated at \( P \). Finally, the orientation of the dumbbell relative to the flow is described, as in §2, by the instantaneous angle \( \alpha \) between the axis of the dumbbell and the \( \xi \) axis. Although we will—for convenience in describing spatial inhomogeneity of the (steady) flow field—use a different local \( (\xi, \psi) \) coordinate system at each time \( t \), it should be emphasized that each such coordinate system is fixed in space and is not regarded as following the motion of the dumbbell. The dumbbell simply happens to pass through the local origin \( P \) at time \( t \).

Rewriting Equations (3) and (4) in terms of the local coordinate system, we now make a Taylor expansion of the fluid velocity field \( \{U(\xi, \psi), V(\xi, \psi)\} \) about \( P \), and compound upon it an expansion in small angle \( \alpha \). The former expansion includes the first correction for inhomogeneity of the flow field—i.e., terms of order \( O(\varepsilon^2) \). The next correction, appearing at order \( O(\varepsilon^4) \), is discarded. Thus, we can approximate the
instantaneous rotation rate of the dumbbell as follows:

\[
\begin{align*}
\hat{\alpha} |_P &= \left\{ \mathcal{V}_\xi + \left[ \frac{1}{6} \mathcal{V}_{\xi\xi\xi} \right] \ell^2 \right\} + \left\{ -2\mathcal{U}_\zeta - \left[ \frac{2}{3} \mathcal{U}_{\xi\zeta\zeta} \right] \ell^2 \right\} \alpha \\
&\quad + \left\{ - [\mathcal{U}_\psi + \mathcal{V}_\zeta] - \left[ \mathcal{U}_{\xi\psi} + \frac{1}{3} \mathcal{V}_{\xi\xi\psi} \right] \ell^2 \right\} \alpha^2 + O(\alpha^3) + O(\ell^4). \\
\end{align*}
\]  

(32)

Here various mixed partial derivatives have been collected together using incompressibility of the fluid in the form

\[
U_\zeta + V_\psi = 0, 
\]  

(33)

which restricts the considerations to two-dimensional flow fields. (In particular, this does not apply within a meridian plane of an axisymmetric flow field.)

Following the motion of the dumbbell's midpoint,

\[
\hat{\xi} |_P = \mathcal{U} + \left\{ \frac{1}{2} \mathcal{U}_{\xi\xi} \ell^2 \right\} + \left\{ \mathcal{V}_{\xi\zeta} \ell^2 \right\} \alpha + \left\{ \frac{1}{2} (\mathcal{U}_{\psi\psi} - \mathcal{U}_{\xi\xi}) \ell^2 \right\} \alpha^2 + O(\ell^2 \alpha^3) + O(\ell^4),
\]  

(34)

\[
\hat{\psi} |_P = \left\{ \frac{1}{2} \mathcal{V}_{\xi\xi} \ell^2 \right\} + \left\{ \mathcal{V}_{\zeta\psi} \ell^2 \right\} \alpha + \left\{ \frac{1}{2} (\mathcal{V}_{\psi\psi} - \mathcal{V}_{\xi\xi}) \ell^2 \right\} \alpha^2 + O(\ell^2 \alpha^3) + O(\ell^4),
\]  

(35)

we consider the instantaneous rate of change of the angle $\beta$ between the fluid velocity vector and the (fixed) local $\xi$ axis. Precisely when the midpoint passes through the local origin $P$, we find

\[
\frac{D\beta}{Dt} |_P = \frac{1}{\mathcal{U}} (\hat{\xi} \mathcal{V}_\xi + \hat{\psi} \mathcal{V}_\psi) 
\]  

(36)

\[
= \left\{ \mathcal{V}_\xi + \frac{1}{2\mathcal{U}} [\mathcal{U}_{\xi\xi} \mathcal{V}_\xi + \mathcal{V}_{\xi\zeta} \mathcal{V}_\psi] \ell^2 \right\} + \left\{ \frac{1}{2\mathcal{U}} [\mathcal{U}_{\zeta\zeta} \mathcal{V}_\xi + \mathcal{V}_{\xi\psi} \mathcal{V}_\psi] \ell^2 \right\} \alpha \\
+ \left\{ \frac{1}{2\mathcal{U}} [\mathcal{U}_{\psi\psi} - \mathcal{U}_{\xi\zeta}] \mathcal{V}_\xi + \mathcal{V}_{\xi\psi} \mathcal{V}_\psi \right\} \ell^2 \alpha^2 + O(\ell^2 \alpha^3) + O(\ell^4). 
\]  

(37)

An observer moving along with the midpoint of the dumbbell will record the following instantaneous rate of change of the dumbbell-streamline angle $\alpha$:

\[
\frac{D\alpha}{Dt} |_P = \dot{\alpha} |_P - \frac{D\beta}{Dt} |_P 
\]  

(38)

\[
= \left\{ \frac{1}{6} \mathcal{V}_{\xi\xi\xi} - \frac{1}{2\mathcal{U}} (\mathcal{U}_{\xi\xi} \mathcal{V}_\xi + \mathcal{V}_{\xi\zeta} \mathcal{V}_\psi) \right\} \ell^2 \\
+ \left\{ -2\mathcal{U}_\zeta \right\} + \left\{ - \frac{2}{3} \mathcal{U}_{\xi\zeta\zeta} - \frac{1}{3\mathcal{U}} (\mathcal{U}_{\xi\xi} \mathcal{V}_\xi + \mathcal{V}_{\xi\psi} \mathcal{V}_\psi) \right\} \ell^2 \alpha \\
+ \left\{ - [\mathcal{U}_\psi - \mathcal{V}_\zeta] - \left[ - \frac{2}{3} \mathcal{U}_{\xi\zeta\zeta} - \frac{1}{3} \mathcal{V}_{\xi\xi\psi} \right] \ell^2 \right\} \alpha^2 + O(\alpha^3) + O(\ell^4) \\
= \{ \mathcal{F}_0 \} \ell^2 + \{ \mathcal{F}_1 \} \ell^2 \alpha + \{ \mathcal{F}_2 \} \ell^2 \alpha^2 + O(\alpha^3) + O(\ell^4). 
\]  

(39)
In general the midpoint of the dumbbell does not move exactly with the corresponding local fluid velocity: both the streamwise and transverse components of the instantaneous (orientation-dependent) migration velocity relative to the fluid scale like $O(\ell^2)$, Equations (34) and (35). [See, e.g., Brunn (1983) in connection with flexible dumbbells.] The resulting motion cannot be predicted a priori, and may even be chaotic (Szeri et al., 1991).

The coefficients $\mathcal{F}_k^{(l)}$ represent specific features of the flow environment that the dumbbell's midpoint samples along its trajectory $[\mathcal{X}^*(t), \mathcal{Y}^*(t)]$ through successive pore throats and bodies, as determined by the coupled nonlinear ODEs (1)–(3):

$$
\mathcal{F}_k^{(l)}(t) = F_k^{(l)}[\mathcal{X}^*(t), \mathcal{Y}^*(t)].
$$

(41)

The superscript "*" is used to emphasize the fact that, although we have listed only the translational degrees of freedom, the angular history is inseparably coupled to them.

Thus far, Equation (40) is merely descriptive on a post facto basis—i.e., after the trajectory in position-orientation space has already been computed. In order for our ODE to be of predictive value, we must somehow decouple the orientational history $\alpha(t)$ from the translational history $[\mathcal{X}^*(t), \mathcal{Y}^*(t)]$ that is needed to compute the coefficients. For this purpose, the general (two-dimensional) formalism developed above is now specialized to the case of pressure-driven flow in a periodically constricted channel, with fore-aft symmetry of the unit cell. The (dimensionless) formula

$$
-H(\alpha X) \leq y \leq H(\alpha X), \quad H(X) = \frac{1}{2} (1 + H_{\text{min}}) + \frac{1}{2} (1 - H_{\text{min}}) \cos(\pi X)
$$

(42)

furnishes a simple example.

We shall indicate with the subscript "f" the pathline (streamline) of a fluid element whose initial position coincides with that of the dumbbell's midpoint. The trajectory $[\mathcal{X}_f(t), \mathcal{Y}_f(t)]$ is a periodically undulating curve. In particular, $\mathcal{X}_f(t)$ and $\mathcal{Y}_f(t)$ are periodic functions of time, the latter having zero mean.

Cumulative displacement of the dumbbell away from the fluid pathline is indicated with the subscript "m", for migration. Thus, we have the decomposition,

$$
\mathcal{X}^*(t) = \mathcal{X}_f(t) + \mathcal{X}_m(t), \quad \mathcal{Y}^*(t) = \mathcal{Y}_f(t) + \mathcal{Y}_m(t),
$$

(43)

with

$$
\mathcal{X}_m(t), \mathcal{Y}_m(t) = O(\ell^2 t).
$$

(44)

For sufficiently short times,

$$
t = O(\ell^{-1}),
$$

(45)

the dumbbell's trajectory follows a fluid pathline within order $O(\ell)$ accuracy. Here we find precisely the translational-rotational decoupling that we desire; but unfortunately the error bound $O(\ell)$ is too crude to be useful in our subsequent derivation, and must be refined.

Writing the counterparts of Equations (34) and (35) for the fixed $(x,y)$ coordinate system, we can approximate the migration velocity by (i) evaluating the relevant derivatives along the fluid pathline instead of the actual trajectory of the dumbbell, and
(ii) neglecting higher corrections due to dumbbell-flow misalignment:

\[ \mathcal{X}_m^*(t) = \ell^2 M[\mathcal{X}_l(t), \mathcal{Y}_{l}(t)] + O(\ell^2) + O(\ell^4), \]
\[ \mathcal{Y}_m^*(t) = \ell^2 N[\mathcal{X}_l(t), \mathcal{Y}_{l}(t)] + O(\ell^2) + O(\ell^4), \]

with

\[ M = \frac{1}{2} \frac{u^2 u_{xx} + 2 v w u_{xy} + v^2 u_{yy}}{u^2 + v^2}, \quad N = \frac{1}{2} \frac{1}{u^2 + v^2} \frac{u^2 v_{xx} + 2 v w v_{xy} + v^2 v_{yy}}{u^2 + v^2}. \]  

Both \( M[\mathcal{X}_l(t), \mathcal{Y}_l(t)] \) and \( N[\mathcal{X}_l(t), \mathcal{Y}_l(t)] \) are periodic functions of time—the latter having zero mean. Thus, the first (orientationally decoupled) correction for migration leads to a periodically undulating trajectory of the dumbbell’s midpoint, indicated here with the subscript “p”,

\[ \mathcal{X}_p(t) = \mathcal{X}_l(t) + \ell^2 \int_0^t M[\mathcal{X}_l(t'), \mathcal{Y}_l(t')] dt', \]
\[ \mathcal{Y}_p(t) = \mathcal{Y}_l(t) + \ell^2 \int_0^t N[\mathcal{X}_l(t'), \mathcal{Y}_l(t')] dt'. \]

In general, no decoupling of translation from the orientational degree of freedom is possible beyond the result,

\[ \mathcal{X}^*(t) = \mathcal{X}_p(t) + O(\ell^2) + O(\ell^4 t^3), \quad \mathcal{Y}^*(t) = \mathcal{Y}_p(t) + O(\ell^2) + O(\ell^4 t^3); \]

but this will be just sufficient.

At this point we assume—subject to \textit{a posteriori} verification—that throughout the “short” time quantified by Equation (45), the dumbbell remains in the nearly—aligned regime,

\[ \alpha = O(\ell). \]

Under this assumption we find

\[ \mathcal{X}^*(t) = \mathcal{X}_p(t) + O(\ell^2), \quad \mathcal{Y}^*(t) = \mathcal{Y}_p(t) + O(\ell^2). \]

In other words, within order \( O(\ell^2) \) accuracy that holds for order \( O(\ell^{-1}) \) times, the dumbbell’s midpoint moves in a periodically undulating trajectory \textit{that is decoupled from the orientational history} \( \alpha(t) \). In evaluating the coefficients \( F^{(1)}_k \) we can replace the actual trajectory \([\mathcal{X}^*(t), \mathcal{Y}^*(t)]\) with the periodic approximation \([\mathcal{X}_p(t), \mathcal{Y}_p(t)]\):

\[ F^{(1)}_k[\mathcal{X}^*(t), \mathcal{Y}^*(t)] = F^{(1)}_k[\mathcal{X}_p(t), \mathcal{Y}_p(t)] + O(\ell^2) \quad \text{for } t = O(\ell^{-1}) \]

Henceforth we shall regard \( \mathcal{Y}_p \) as a function of \( \mathcal{X}_p \) along the trajectory. In view of Equations (52) and (54), the orientational evolution equation (40) becomes

\[ \frac{d\alpha}{dt} = F^{(2)}_0[\mathcal{X}_p(t), \mathcal{Y}_p[\mathcal{X}_p(t)]] \ell^2 \alpha + F^{(1)}_1[\mathcal{X}_p(t), \mathcal{Y}_p[\mathcal{X}_p(t)]] \alpha \]
\[ + F^{(1)}_2[\mathcal{X}_p(t), \mathcal{Y}_p[\mathcal{X}_p(t)]] \ell^2 \alpha + O(\ell^6) \quad \text{for } t = O(\ell^{-1}). \]

Now that longitudinal position \( \mathcal{X}_p(t) \) is completely decoupled from \( \alpha(t) \), we can make a coordinate transformation whereby longitudinal position \( x \) becomes the new inde-
pendent variable in place of time $t$:

$$
\frac{dx}{dt} = \frac{1}{u[x, \mathcal{Y}_p(x)] + \ell^2 M[x, \mathcal{Y}_p(x)] \frac{dx}{dt}}
$$

Thus, we find

$$
\frac{dx}{dt} = \ell^2 F^{[2]}_0[x, \mathcal{Y}_p(x)] + F^{[1]}_1[x, \mathcal{Y}_p(x)] + F^{[0]}_2[x, \mathcal{Y}_p(x)] a^2 + O(\ell^3) \quad \text{for } x = O(\ell^{-1}),
$$

with

$$
F^{[2]}_0(x, y) = \frac{1}{u} \left[ \frac{1}{6} \varphi^{\xi\xi\xi} - \frac{1}{2} \mathcal{Y}(\varphi_{\xi\xi}) \varphi_{\xi} \right],
$$

$$
F^{[1]}_1(x, y) = -\frac{\partial \mathcal{Y}}{\partial x},
$$

$$
F^{[0]}_2(x, y) = -\frac{1}{u} [\mathcal{Y}_{\phi} + \varphi_{\phi}].
$$

The small-angle assumption (52) must be validated a posteriori from the solution $\alpha(x)$. To be precise, we mean the following. The governing Equation (58) was derived for $x = O(\ell^{-1})$, under the assumption that $\alpha = O(\ell)$ for $x$ so restricted. If we start off with the dumbbell perfectly aligned with the flow, $\alpha(0) = 0$, then we can use Equation (58) to determine $\alpha(x)$ as long as $\alpha$ remains of order $O(\ell)$—as determined from the leading-order asymptotic solution, Equations (72) and (73) to be derived below in §5. In particular, we shall see that nothing was "wasted" in the error bounds: as $x$ increases, neither condition cuts off the other prematurely—in the rough sense of asymptotic order. Note that the normalization constant of the flow field cancels out in Equations (59)–(61).

A troubling observation: the functions $F^{[1]}_k[x, \mathcal{Y}_p(x)]$ do not in general depend periodically upon $x$. To see this we must characterize the difference between the fluid pathline $y = \mathcal{Y}_f(x)$ and the dumbbell's approximate trajectory $y = \mathcal{Y}_p(x)$. Both curves undulate periodically,

$$
\mathcal{Y}_f(x + nL_f) = \mathcal{Y}_f(x), \quad \mathcal{Y}_p(x + nL_p) = \mathcal{Y}_p(x) \quad (n = \pm 1, \pm 2, \pm 3, \ldots),
$$

but it follows from Equation (49) that their periods must in general differ:

$$
L_p(\ell) - L_f = \text{ord}(\ell^2).
$$

With a suitable stretching of the $x$ coordinate, both curves can be made to fall on top of each other within order $O(\ell^2)$ accuracy:

$$
\mathcal{Y}_p(x) - \mathcal{Y}_f(xL_p/L_f) = O(\ell^2) \quad (0 \leq x < \infty).
$$

As $x$ increases, the dumbbell's approximate trajectory $y = \mathcal{Y}_p(x)$ gradually becomes shifted in phase relative to the fluid streamline $y = \mathcal{Y}_f(x)$, the latter being tied to the periodically corrugated geometry. Exact periodicity of the functions $F^{[1]}_k[x, \mathcal{Y}_p(x)]$ is
lost unless the ratio $L_2\ell_1 / L_2\ell$ is a rational number. Even in that case the ord($\ell^{-2}$) period—i.e., the lengthscale of “local” structure over which one would take the averages in the subsequent two-scale analysis (§5, below)—is actually longer than the ord ($\ell^{-1}$) flipping distance that we seek to explain.

In rewriting Equation (58) using $\gamma_l(x)$ in place of $\gamma_p(x)$, we cannot, in general, claim a better error estimate than

$$\gamma_p(x) - \gamma_l(x) = O(\ell) \quad \text{for} \quad x = O(\ell^{-1}). \quad (65)$$

This approximation leads to order $O(\ell^3)$ errors in the terms $\ell^2 F_1^{[2]}$ and $F_2^{[1]} x^2$, whose coefficients now vary periodically with $x$:

$$F_0^{[1]}[x, \gamma_p(x)] = F_0^{[1]}[x, \gamma_l(x)] + O(\ell) \quad \text{for} \quad x = O(\ell^{-1}). \quad (66)$$

But the $F_1^{[1]} x$ term poses an essential problem in Equation (58), because with the same substitution we must be prepared for errors of order $O(\ell^2)$—which need not be small compared with either $\ell^2 F_1^{[2]}$ or $F_2^{[1]} x^2$. Both of these terms will be seen in §5 to be crucial for explaining the ord($\ell^{-1}$) flipping distance. Since it is physically untenable to keep some terms while neglecting others of the same magnitude, we cannot, in general, proceed further.

The above difficulty does not arise, however, in three special cases: (i) $F_1^{[1]}$ does not depend upon $y$ at all, whereby the shape of the trajectory $y = \gamma_p(x)$ has no bearing upon $F_1^{[1]}(x)$; (ii) $F_1^{[1]}$ depends sufficiently weakly upon $y$ that the effect of the difference between $\gamma_p(x)$ and $\gamma_l(x)$ is negligible compared with the other terms; and/or (iii) one can find one (or more) streamline(s) along which the time-periodic migration velocity $M[\gamma_l(t), \gamma_l(t)]$ has zero mean, whereby the periods of $\gamma_p(x)$ and $\gamma_l(x)$ exactly match for any value of $\ell$. Fortunately, cases (i) and (ii) arise in using lubrication theory to approximate the flow field (§6, below), which means the results to be obtained will apply for any streamline. [The question of being able to find special streamlines for case (iii) will be considered in a subsequent paper.] Thus, we finally obtain

$$\frac{dz}{dx} = \ell^2 a(x) + b(x) + c(x) x^2 + O(\ell^3) \quad \text{for} \quad x = O(\ell^{-1}), \quad (67)$$

with the periodic coefficients

$$a(x) = F_1^{[2]}[x, \gamma_l(x)], \quad b(x) = F_1^{[1]}[x, \gamma_l(x)], \quad c(x) = F_2^{[1]}[x, \gamma_l(x)]. \quad (68)$$

Note that $b(x)$ must have zero mean, owing to Equation (60) and fore-aft symmetry of the unit cell. The term $\ell^2 a(x)$ represents the crucial influence of spatial inhomogeneity of the flow field: it is responsible for driving the dumbbell out of alignment with the local streamlines—as is immediately obvious when the dumbbell starts out perfectly aligned ($x = 0$).

5 A TWO-SCALE ANALYSIS FOR THE NEARLY-ALIGNED REGIME

In considering the rotary motion of a freely-suspended, freely-draining dumbbell (dimensionless length $2\ell$) relative to the surrounding streamlines in a periodically corrugated channel, developments in the previous section led to an ODE (67) of the
Riccati type (Ince, 1956, §2.15) with periodic coefficients. As this equation applies only for small $\alpha$, Equation (52), we cannot address the actual flipping-over of the dumbbell. But the apparent $\ell^{-1}$ scaling of the distance traveled between flips, which was observed numerically in §2, can be explained by a two-scale analysis of the incipient stages of misalignment, corresponding to the outer solution in §3. Proceeding along analogous lines—using $\xi = x$ and $X = \ell x$ as the short and long longitudinal coordinates, respectively—we find

$$a_0(\xi, X) \equiv 0, \quad a_1(\xi, X) = S(X)e^{B(\xi)}, \quad (69)$$

where

$$B(\xi) = \int_0^\xi b(\xi')d\xi' \quad (70)$$

is periodic because $b(\xi)$ has zero mean. To avoid secular terms at second order we must require that

$$S'(X) = \langle a(\xi)e^{-B(\xi)} \rangle + \langle c(\xi)e^{B(\xi)} \rangle \langle S(X) \rangle^2 = \bar{a} + \bar{c}[S(X)]^2. \quad (71)$$

Provided that $\bar{a} \neq 0$, the long lengthscale is of order $\ell^{-1}$ and not $\ell^{-2}$ as one might initially expect from Equation (67). At leading order we find for the solution,

$$a(x; \ell) \sim \ell(\bar{a}/\bar{c})^{1/2}e^{B(x)} \tan(x\ell/\sqrt{\bar{a}\bar{c}}) + O(\ell^2). \quad (72)$$

The asymptotic solution blows up at $x = (\pi/2)(\bar{a}/\bar{c})^{-1/2}/\ell^{-1}$. This asymptotic behavior is suggestive (but certainly no rigorous proof) of a movable singularity—i.e., a singularity whose location is determined by the initial condition; see Ince (1956, §§12.5–12.52). To avoid the singularity we must be more specific about the scaling constant in the bound (67):

$$0 \leq x \leq \left\{\bar{c}/\sqrt{\bar{a}\bar{c}}\right\} \ell^{-1}, \quad \bar{c} < \pi/2. \quad (73)$$

Under this condition the asymptotic solution stays within the small-angle regime defined by Equation (52). We cannot then model the actual flipping-over of the dumbbell, but the long ord($\ell^{-1}$) lengthscale certainly seems to explain the numerically observed scaling of the distance traveled between successive flips (§2).

6 A LUBRICATION APPROXIMATION

For an illustrative calculation using the asymptotic solution formula (72), we seek to approximate the coefficients $\bar{a}$ and $\bar{c}$ and the function $B(x)$ in analytical form. Thus we shall consider the case of long wavelength $\lambda = 2/\varepsilon$ of the wall corrugations described by Equation (42), which means $\varepsilon \ll 1$. As the lubrication theory for this problem follows along standard lines—see e.g., Leal (1992, chap. 7)—we omit the derivations and list only the relevant results. Normalized for unit areal flowrate (integrated over $y$), the flow field is given by

$$u(x, y) \sim u_{1,0}[ex, Y(ex, y)] + \varepsilon^2 u_{1,2}[ex, Y(ex, y)] + O(\varepsilon^4), \quad (74)$$

$$v(x, y) \sim \varepsilon v_{1,1}[ex, Y(ex, y)] + \varepsilon^3 v_{1,3}[ex, Y(ex, y)] + O(\varepsilon^5), \quad (75)$$
with

\[ u_{L0}(X,Y) = \frac{3}{4} \frac{1}{H(X)} (1 - Y^2), \]  
\[ u_{L2}(X,Y) = \frac{3}{40} \left( 4[H(X)]^{-1} [H'(X)]^2 - H''(X) \right) (1 - Y^2)(1 - 5Y^2), \]  
\[ v_{L1}(X,Y) = \frac{3}{4} \frac{H'(X)}{H(X)} Y(1 - Y^2), \]  
\[ v_{L3}(X,Y) = \frac{3}{40} \left( Y(1 - Y^2) [4[H(X)]^{-1} [H'(X)]^3(1 - 5Y^2) \right. \]  
\[ - 4H'(X)H''(X)(2 - 3Y^2) + H(X)H'''(X)(1 - Y^2) \right). \]

Here we have defined a stretched longitudinal coordinate and a modified transverse coordinate:

\[ X = \varepsilon x, \quad Y(X,y) = y/H(X). \]  

Substituting these expressions into the coefficient formulas (59)–(61), and regarding \( X \) as the independent variable instead of \( x \) in Equation (58), we obtain a new orientational evolution equation,

\[ \frac{d\tilde{x}}{dX} = \varepsilon^2 \varepsilon^3 \tilde{F}_{0,1,0}^{(2)}[X,\Theta_{f,1,2}(X)] + \tilde{F}_{1,1,2}^{(0)}[X,\Theta_{f,1,2}(X)] \tilde{x} + \varepsilon^{-1} \tilde{F}_{2,1,0}^{(0)}[X,\Theta_{f,1,2}(X)] \tilde{x}^2 \]  
\[ + O(\varepsilon^4 \varepsilon^2) + O(\varepsilon^5 \varepsilon^2) + O(\varepsilon^6 \varepsilon) + O(\varepsilon^7) \quad \text{for} \quad X = O(\varepsilon^{-1} \varepsilon^{-1}), \]

where the subscript "f,1,2" means that the approximate fluid streamline \( y = \Theta_{f,1,2}(X) \) is obtained from the two-term lubrication approximation appearing in Equations (74) and (75). Note that the domain of \( X \) extends over many wavelengths of the wall corrugations. The reduced coefficients are given by

\[ \tilde{F}_{0,1,0}^{(2)}(X,y) = Y(X,y) a_{L0}(X), \]  
\[ \tilde{F}_{1,1,2}^{(0)}(X,y) = b_{L0}(X) + \varepsilon^2 b_{L2}^{(1)}(X)(1 - 3[Y(X,y)]^2) \]  
\[ + b_{L2}^{(2)}(X)(7 - 11[Y(X,y)]^2), \]  
\[ \tilde{F}_{2,1,0}^{(0)}(X,y) = Y(X,y)c_{L0}(X)/(1 - [Y(X,y)]^2), \]

with

\[ a_{L0}(X) = \frac{1}{6} \left\{ \frac{H'''}{H} - 7 \frac{H''H'}{H} - 3 \frac{(H')^2}{H} + 15 \frac{H''(H')^2}{H^2} \right\}, \]  
\[ b_{L0}(X) = 2H'(X)/H(X), \]  
\[ b_{L2}^{(1)}(X) = H(X)H'''(X)/5, \quad b_{L2}^{(2)}(X) = -H'(X)H''(X)/5, \]  
\[ c_{L0}(X) = 2/H(X). \]
The error bounds in Equation (81) are based upon the result

$$\Psi_p(X) = \Psi(X) + O(\varepsilon \ell) = \Psi_{f,1,2}(X) + O(\varepsilon \ell) + O(\varepsilon^3 \ell^{-1}) \quad \text{for} \quad X = O(\varepsilon^{-1} \ell^{-1}),$$

(89)

which follows from Equations (48)-(50), (74) and (75). Subject to a posteriori verification using the asymptotic solution, we have also assumed that

$$\alpha(X;\varepsilon,\ell) = O(\varepsilon^2 \ell) \quad \text{for} \quad X = O(\varepsilon^{-1} \ell^{-1});$$

(90)

see relevant discussion appearing immediately after Equation (61), above. Note that we must retain the first lubrication correction beyond leading order in $\tilde{F}_{0,1,2}$, Equations (83), (86) and (87), whereas the leading-order results suffice in $\tilde{F}_{0,1,0}^{[2]}$ and $\tilde{F}_{2,1,0}^{[0]}$.

At leading order the fluid streamlines $y = \Psi_{f,1,0}(X)$ are given by $Y(X,y) = \text{constant}$. At this crudest level of approximation, $Y$ functions only as the parameter that indicates on which streamline the dumbbell starts, and one can discard the $\varepsilon^2$ correction in Equation (83). Nothing that

$$\Psi_{f,1,0}(X) = Y_0 H(X) + O(\varepsilon \ell^{-1}),$$

(91)

we find the simpler orientational evolutional equation,

$$\frac{d\alpha}{dX} = \varepsilon^2 \ell \tilde{F}_{0,1,0}^{[2]}[X,Y_0 H(X)] + b_{1,0}(X)\alpha + \varepsilon^{-1} \tilde{F}_{2,1,0}^{[0]}[X,Y_0 H(X)]\alpha^2$$

$$+ O(\varepsilon^4 \ell^4) \quad \text{for} \quad X = O(\varepsilon^{-1} \ell^{-1}).$$

(92)

In order that the crucial $\varepsilon^2 \ell^3 \tilde{F}_{0,1,0}^{[2]}$ term dominates over the neglected terms in Equation (92) we must require that $\varepsilon \ll \ell$. The more complicated orientational evolution equation (81) is less restrictive: we can, for example, take $\varepsilon \sim \ell$. We shall, however, use the cruder version for illustrative calculations, because it is much simpler to regard $Y$ as a constant parameter along the fluid streamline. In Figure 6 the coefficients $a_{1,0}$, $b_{1,0}$ and $c_{1,0}$ are plotted against $X$ for three values of the constriction ratio of the channel ($H_{\min} = 1/4, 1/2, 3/4$), Equation (42).

As applied to Equation (92), the asymptotic solution (72) can be written in the following form:

$$\alpha(X;\varepsilon,\ell) \sim \varepsilon^2 \ell [(1 - Y^2) \tilde{a}_{1,0}/\tilde{c}_{1,0}]^{1/2} \left[ \frac{H(X)}{H(0)} \right]^2 \tan \left\{ X \varepsilon \ell Y (\tilde{a}_{1,0}/\tilde{c}_{1,0})(1 - Y^2) \right\}^{1/2},$$

(93)

with

$$\tilde{a}_{1,0} = \left\langle a_{1,0}(X) \left[ \frac{H(0)}{H(X)} \right]^2 \right\rangle, \quad \tilde{c}_{1,0} = \left\langle c_{1,0}(X) \left[ \frac{H(X)}{H(0)} \right]^2 \right\rangle = 1 + H_{\min}.$$

(94)

Figure 7 shows $\tilde{a}_{1,0}$ as a function of $H_{\min}$. The important feature with regard to the two-scale asymptotics is that $\tilde{a}_{1,0}$ does not vanish, whereby the flipping scales inversely with the length of the dumbbell when the pore geometry is held fixed.

Figure 8 compares the asymptotic solution (93) with the numerics from §2. We have chosen $H_{\min} = 0.6$ and $\lambda = 2$ in order to match the two-dimensional channel to the meridian-plane generator of the axisymmetric pore from Figure 1. Here we make only a qualitative comparison; our asymptotic analysis does not apply to axisymmetric flow.
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[Note, however, that leading-order lubrication theory does give the same functional form for the streamlines in both geometries.] One expects that such a short wavelength will severely strain the accuracy of the lubrication approximation. Furthermore, in duplicating the length of the dumbbell considered in Figure 3c (2\(\epsilon\) = 0.1), we must recognize that the dumbbell is not very short compared with the pore scale. It should also be pointed out that our example violates the aforementioned condition \(\epsilon \ll \epsilon\). Finally, the asymptotic curve may extend beyond the realms of "short" distances of

![Graphs showing coefficients as functions of X]

FIGURE 6 Coefficients \(a_{L0}\), \(b_{L0}\) and \(c_{L0}\) from Equations (85), (86) and (88), plotted as functions of stretched axial position \(X\) for three constriction ratios \(H_{\text{min}} = 1/4, 1/2, 3/4\) of the sinusoidally corrugated channel.
travel and "small" angles of misalignment; cf. Equation (90). Having mentioned these caveats, we note that the asymptotic and numerical curves appear strikingly similar. Evidently, the Riccati equation (67) captures the essential features of the orientational dynamics.

FIGURE 6 (Continued).

FIGURE 7 Averaged coefficient $\hat{a}_{lo}$ from Equation (94) plotted as a function of the constriction ratio $H_{\min}$. 
FIGURE 8  Asymptotic formula (93) for a two dimensional channel [Equation (42)] compared with the numerical solution of (2) from §2 (Figure 3) for an axisymmetric pore (Figure 1). Here we consider matching values of the constriction ratio \( H_{\min} = R_{\min} = 0.6 \) and wavelength \( \lambda = 2 \) of the wall corrugations in the two geometries. The dumbbells have the same length \( (2L = 0.1) \) and start perfectly aligned with the flow \( \alpha = 0 \) at analogous positions \( (x = 0, y = 0.3) \) along corresponding streamlines.

7 CONCLUDING REMARKS

The fluctuation-flipping behavior of freely-draining dumbbells in converging-diverging flow fields was originally observed numerically (Nitsche, 1996) using a flexible connection between the beads. This behavior has now been explained here asymptotically for the simpler model of a rigid connection. Deterministic cross-stream migration in the former case arose from the combination of (i) the resulting misalignment of the dumbbell relative to the local streamlines, and (ii) flow-deformation nonlinearities associated with flexible dumbbells; see also Brunn (1983). Given the predominance of the freely-draining assumption in kinetic theory of polymers, it is important to point out that the deterministic flipping mechanism analyzed above is not present in the inhomogeneous flow fields that are usually used to model polymer migration effects: (i) rectilinear tube or channel flow; and (ii) rotary Couette flow between coaxial cylinders. Only with higher-order hydrodynamic interactions does the local shear in these simpler flow fields cause Jeffery-orbit tumbling and resultant dumbbell-streamline misalignment (Brunn, 1983).

Even for the simplest conceivable model of a polymer (freely-draining rigid dumbbell) in a crude representation of the converging-diverging flow fields in porous media (sinusoidally corrugated tube or channel), deterministic convection is seen here to possess an unexpectedly rich structure. Spatial inhomogeneity of the (steady) flow field embodies two crucial elements necessary for fluctuation-flipping orbits: (i) motion between different local flow environments; and (ii) finite-size effects requiring higher corrections to the local, linearized velocity field. The former aspect resides within the
dynamical-systems framework of Szeri et al. (1991, 1992) and Szeri and Leal (1993), but the latter has there been neglected.

In considering elastic dumbbells, many passages through successive pore throats and bodies have a cumulative effect, to which one cannot apply the concept of local migration velocity as a function of local measures of flow inhomogeneity. Particularly because most proposed mechanisms of polymer migration predict relatively weak effects (Brunn, 1983; Agarwal et al., 1994), the above rotary transport process could represent a significant convective input to the kinetic theory.

Finally, we remark that the fluctuation-flipping orbits analyzed here contribute to a growing conceptual "database" on translational-rotational dynamics of rigid particles or conformational dynamics of polymers— as well as resultant transport properties and collective rheological behavior—in porous media (Nollert and Olbricht, 1985; Shaqfeh and Koch, 1988i, ii, 1992; Frattini et al., 1991; Evans et al., 1994) and suspensions (Brenner, 1974; Koch and Shaqfeh, 1990; Shaqfeh and Koch, 1990; Rahnama et al., 1993, 1995; Harlen and Koch, 1992, 1993). Our conclusions appear particularly interesting in comparison with the result of Shaqfeh and Koch (1988i)—also in the absence of Brownian motion—that a rodlike particle should become aligned with the flow that carries it through a dilute, random bed of spheres or fibers. Their angular velocity was based upon the local linearized velocity field, and yielded perfect alignment of infinitesimally thin rods. In our analysis, a "nonlocal" [cf. Koch and Brady (1987) in a different but related context] type of correction is seen to cause misalignment of such a body during flow through a periodically corrugated pore or channel. One common element can, however, be found: in both cases the appreciable net effect (alignment vs misalignment) accumulates over many successive, weak "interactions"
with individual "elements" of the porous microstructure. In the paper of Shaqebe and Koch (1988i), each "hydrodynamic collision" with a bed particle produced a small change in orientation; the random nature of these small fluctuations (due to the random positions of particles in the bed) led to a diffusive process on a longer timescale. Here each passage through one wavelength of the periodic wall corrugations leads to a small change in the dumbbell-streamline angle, and the asymptotic solution shows deterministic behavior.

NOMENCLATURE

\( a \quad (1.) \text{Exponent describing overlap region between inner and outer solutions (§3); or} \\
\text{(2.) Coefficient function in Riccati equation (§§4–6)} \)

\( A \quad \text{Slowly-varying function appearing in the two-time expansion (§3)} \)

\( b \quad (1.) \text{Exponent describing overlap region between inner and outer solutions (§3); or} \\
\text{(2.) Coefficient function in Riccati equation (§§4–6)} \)

\( B \quad (1.) \text{Slowly-varying function appearing in the two-time expansion (§3); or} \\
\text{(2.) Antiderivative of } b \text{ (§5)} \)

\( c \quad \text{Coefficient function in Riccati equation (§§4–6)} \)

\( C \quad \text{Constant in bounds on } t, x \text{ and } X \text{ (§§3, 5, 6)} \)

\( f \quad \text{Function describing rotary fluctuations in the general case (§3)} \)

\( F \quad (1.) \text{Antiderivative of } f \text{ (§3); or} \\
\text{(2.) Coefficient functions describing the inhomogeneous flow environment} \\
through which the dumbbell passes, } F = F(x,y) \text{ (§§4, 6)} \)

\( \mathcal{F} \quad \text{Coefficient functions describing the progression of local flow environments} \\
sampled by the dumbbell's midpoint, } \mathcal{F}(t) = F[\mathcal{F}^*(t),\mathcal{F}^*(t)] \text{ (§4)} \)

\( H \quad \text{Function describing the shape of wall corrugations (§§4, 6)} \)

\( H_{\text{min}} \quad \text{Constriction ratio of the sinusoidally corrugated 2-d channel (§§4, 6)} \)

\( \ell \quad \text{Dimensionless half-length of rigid dumbbell (§§2, 4–6)} \)

\( L \quad \text{Period of a periodically undulating trajectory (§4)} \)

\( M, N \quad x \text{ and } y \text{ components of migration velocity (§4)} \)

\( P \quad \text{Origin of local } (\xi, \psi) \text{ Cartesian coordinates, coincident with the midpoint of} \\
\text{the dumbbell at a particular time (§4)} \)

\( R \quad \text{Axis ratio (minor/major) of a prolate spheroidal particle (§3)} \)

\( R_{\text{min}} \quad \text{Constriction ratio of sinusoidally corrugated axisymmetric pore (§2)} \)

\( S \quad \text{Function varying on the long lengthscale in the two-scale expansion (§5)} \)

\( t \quad \text{Dimensionless time (§§2–4)} \)

\( \delta t \quad \text{Time step in fourth-order Runge-Kutta scheme (§§2, 3)} \)

\( T \quad \text{Slow time variable in the two-time analysis, } T = st \text{ (§3)} \)

\( u, v \quad x \text{ and } y \text{ components of the fluid velocity (§§2, 4, 6)} \)

\( U, V \quad \xi \text{ and } \psi \text{ components of velocity, referred to a local, flow-oriented coordinate} \\
system (§4) \)

\( \xi, \psi \quad \xi \text{ and } \psi \text{ components of velocity evaluated at the origin } P \text{ of the local flow-} \\
\text{oriented coordinate system (§4)} \)
Dimensionless Cartesian coordinates in the meridian plane of an axisymmetric flow or in a two-dimensional flow ($\S\S 2.4-6$)

$x, y$  

Stretched $x$ coordinate, $X = \varepsilon x$ ($\S\S 5, 6$)

$Y$  

Reduced $y$ coordinate in the corrugated channel, $Y = y/H(x) = y/H(X)$ ($\S 6$)

$\xi$  

Lagrangian $x$ coordinate of dumbbell's midpoint ($\S\S 2, 4$)

$\psi$  

(1.) Lagrangian $y$ coordinate of dumbbell's midpoint ($\S\S 2, 4$)

(2.) Function indicating the shape of a trajectory, $y = \psi(x)$ ($\S\S 4, 6$)

Greek letters

$\alpha$  

Angle between dumbbell axis and the fluid velocity vector at its midpoint ($\S\S 2.4-6$)

$\beta$  

Angle between the fluid velocity vector and the local $\xi$ axis ($\S 4$)

$\varepsilon$  

(1.) Reduced amplitude of harmonic rotary fluctuation velocity superimposed upon steady, homogeneous shear ($\S 3$); or

(2.) Longitudinal stretching parameter in lubrication theory ($\S\S 4, 6$)

$\phi$  

Angle of dumbbell axis measured relative to the $x$ axis ($\S\S 2, 3$)

$\gamma$  

Shear rate ($\S 3$)

$\lambda$  

Wavelength of wall corrugations of axisymmetric pore or two-dimensional channel ($\S\S 2.6$)

$\tau$  

Fast time variable in two-time analysis, $\tau = t$ ($\S 3$)

$\xi, \psi$  

Local flow-oriented Cartesian coordinates ($\S 4$)

$\Xi, \Psi$  

Lagrangian $\xi$ and $\psi$ coordinates of dumbbell's midpoint ($\S 4$)

Superscripts

(I)  

Indicates inner region or asymptotic solution ($\S 3$)

$[.]$  

Indicates power of $\varepsilon$ for coefficients $F$ and $f$ ($\S 4$)

(M)  

Indicates common asymptotic behavior for inner-outer matching in the region of overlap ($\S 3$)

(O)  

Indicates outer region or asymptotic solution ($\S 3$)

(U)  

Indicates uniformly valid asymptotic approximation ($\S 3$)

*  

Indicates coupling between translational and rotational motion, and resultant history dependence ($\S 4$)

Subscripts

$f$  

Indicates fluid pathline (streamline) ($\S 4$)

$f, L2$  

Indicates fluid streamline computed using the two-term lubrication approximation ($\S 6$)

$i$  

Index of terms in two-time expansion ($\S 3$)

$k$  

Indicates power of $\varepsilon$ for coefficients $F$ and $f$ ($\S 4$)

$L_i$  

Indicates the power of $\varepsilon$ in the lubrication approximation ($\S 6$)

$m$  

Indicates migration of the dumbbell relative to the fluid velocity vector at its midpoint ($\S 4$)

$p$  

Indicates (periodic) approximate trajectory of the dumbbell's midpoint incorporating the first (orientationally decoupled) correction for migration relative to the fluid ($\S 4$)
x, y  Indicate partial differentiation with respect to x and y, respectively (§4)
\(\xi, \psi\)  Indicate partial differentiation with respect to \(\xi\) and \(\psi\), respectively (§4)

Marks over symbols

.  Indicates time derivative (§§2, 4)
–  Indicates weighted average, according to Eq. (71) or (94) (§§5, 6)
\(\sim\)  Indicates reduced versions of the coefficient functions \(F\), obtained from lubrication theory

Other

\langle \rangle  Average of a periodic function over one period (§§3, 5, 6)

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