Introduction to Error Calculation and Propagation

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1 Introduction

Every time we measure a physical quantity, that quantity has a certain error or uncertainty. This is because our instruments and methods are never a hundred percent perfect and we might make small mistakes here and there. Also the environment in which we take our measurements may vary from one reading to another (e.g. temperature variations). Based on the origin of the error, we differentiate two basic types of errors, systematic and random errors.

1.1 Accuracy and Precision

In Science we differentiate between accuracy and precision and it is important that you know the difference and also use these terms when discussing your results in your lab reports. When we measure a specific quantity during an experiment, we will never measure its true value as there are always some measurement errors involved that cause flaws in our results. Accuracy is now defined as the closeness of our measurements to the "true" value. We might get a slightly different value after each measurement, but at the end all our values scatter around the "true" value of the quantity we are measuring. Such an experiment is described as accurate. This obviously means we know the "true" value from theory or literature. If in addition to that, every repeat of the measurement gives us more or less the same result, so that means all our measurements cluster closely around a value, then our experiment is also precise. You can have four different cases now (figure 1).

a. Our measurements cluster closely around the "true" value → *accurate and precise.*
b. Our measurements cluster not very closely around the "true" value → *accurate but not precise.*
c. Our measurements cluster closely around a value that is not the "true" value → *precise but not accurate.*
d. Our measurements cluster not very closely around a value that is not the "true" value → *not accurate and not precise.*

1.2 Systematic Errors

Systematic errors are errors that are constant throughout the experiment. That means all readings that you take will have the same systematic error and will either be too big or too small compared to the true value. Systematic errors originate from inexact methods and faulty instruments. An example would be an ammeter (i.e. instrument to measure the electric current in a circuit) that is not calibrated properly. Another common sys-

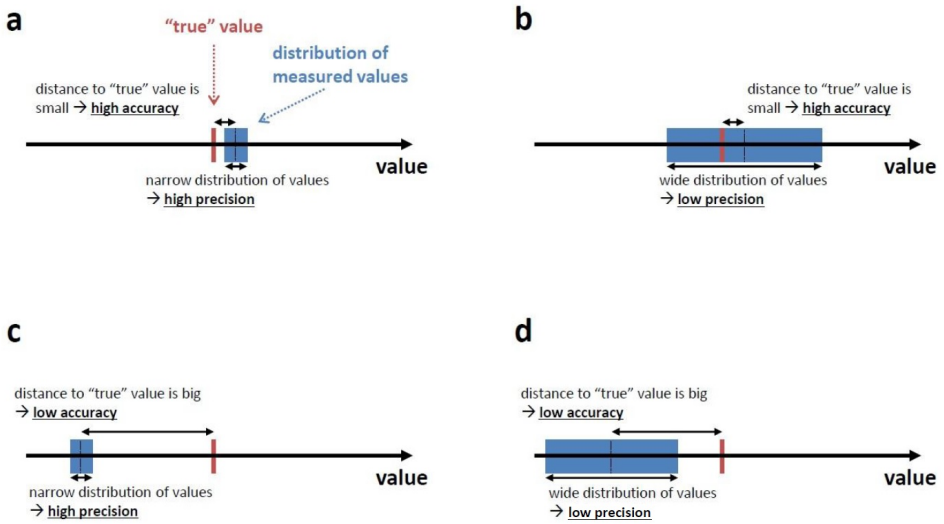

Figure 1: a: Accurate and precise. b: Accurate but not precise. c: Precise but not accurate. d: Not accurate and not precise.

tematic error is the parallax error. This is a human error that occurs when reading measurements off a scale under a certain angle. That angle causes the readings to be slightly off. Another human error would be reaction time, for example when using a stopwatch to record the time it takes a certain object to travel a defined distance. Observer with a fast reaction time will inevitably record readings that are closer to the true value than observer with a slow reaction time. However, a systematic error is characterised in a sense that all readings are corrupted in the same way. We won't consider systematic errors any further as they can not be accounted for with mathematical statistics. Therefore, if a systematic error is identified in your experiment, you should mention it in the discussion section of your lab report. Furthermore, make some suggestions how you can minimise that error if you were to do the experiment again.

1.3 Random Errors

Random errors can occur through a variety of factors and do not follow a pattern such as the systematic errors. That means random errors corrupt your readings in a completely random and unpredictable fashion. Random errors can be averaged out by taking a large number of readings. It is for that reason that researchers usually try to conduct experiments with a large sample group, whether it be a large group of volunteers for a medical study or engineers that take many repeat readings off an instrument. But keep in mind that the expected outcome of an experiment should always justify the efforts you put into it. Examples of random errors are certain instrumental

problems that occur even when using well calibrated and high quality equipment, changes in the environment of your experiment such as temperature, pressure, and humidity. Further examples are mechanical vibrations and the interference of electrical and magnetic fields with your experiment. Because these errors are completely random and unpredictable, we use statistics to account for them.

The following sections of this handout will now try to give you an overview of important concepts that will help you during the write up of your lab reports. When you have worked through this handout, we want you to

- be able to analyse a series of readings by calculating the **mean** and the **standard deviation**
- explain how **errors propagate** if you use one or several quantities that have errors associated with them to calculate another quantity, and
- be able to determine a **regression curve** that gives a functional relation between two physical quantities.

But before we really start wrapping our heads around these concepts, we want to familiarise ourselves with what mathematicians and scientists call a histogram and a normal distribution.

1.4 Statistical Distribution of Measurements

We will begin this section with an imaginary experiment. Let us assume we want to measure the time it takes a tennis ball to fall from a rooftop to the ground. If we repeat the experiment over and over we end up with a series of measurements (in our case time measurements) x_1, x_2, ... , x_n. If we want to take a look at the distribution of these measurements, we first sort them according to their values and then introduce classes. When plotting this graphically, this will lead to what is known as a histogram (figure 2). Let's say we have measured the time it takes our tennis ball to drop to the ground exactly 80 times and we received values between 1.0 and 2.0 seconds. We will sort these values now in 10 classes, each of a width of 0.1 seconds (table 1). When we plot the classes on the abscissa and the occurrence on the ordinate, we receive a histogram as a graphical representation of our data (figure 2).
If we now increase the number of readings by a lot ($n \to \infty$) and at the same time we make the class size Δ x smaller and smaller ($\Delta x \to 0$), we will receive a continuous distribution. This continuous distribution is described by a function f(x) called a distribution density function (figure 3).

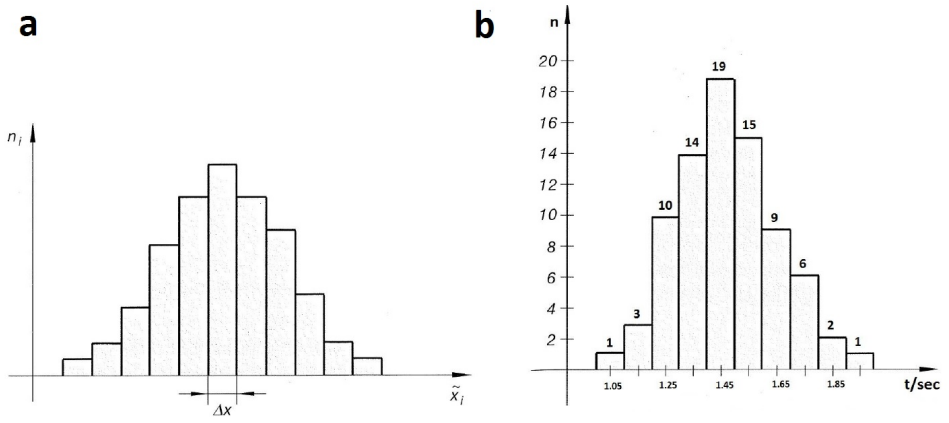

Figure 2: a: Depiction of a standard histogram (bar graph). b: Histogram of our tennis ball experiment.

Table 1: Tennis Ball Experiment

Class No i	Class Borders (sec)	Centre of Class (sec)	Occurrence
1	1.0...1.1	1.05	1
2	1.1...1.2	1.15	3
3	1.2...1.3	1.25	10
4	1.3...1.4	1.35	14
5	1.4...1.5	1.45	19
6	1.5...1.6	1.55	15
7	1.6...1.7	1.65	9
8	1.7...1.8	1.75	6
9	1.8...1.9	1.85	2
10	1.9...2.0	1.95	1

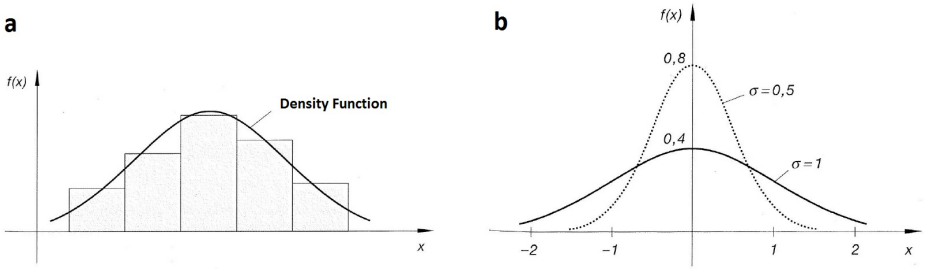

Figure 3: a: Transition from a discrete distribution (i.e. histogram) to a continuous distribution. b: Standard deviation σ determines height and width of the normal distribution.

From figure 3 you can see that the readings are distributed symmetrically around a maximum. Furthermore, the bigger the deviation of a certain reading from the maximum, the less likely is its occurrence. This function was first described by the German mathematician Carl Friedrich Gauß (1777 - 1855) and is called a normal distribution.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

This function is bell shaped and the parameters μ and σ are called mean and standard deviation, respectively. If we would take an infinite number of readings, then the parameter μ is the value that is most likely to occur as it is the absolute maximum of our normal distribution. The standard deviation σ determines height and width of our density function as is shown in figure 3. The smaller σ is, the more pronounced is the maximum. The standard deviation σ therefore describes the mean variation of our readings x_i around the "true" but unknown value μ. We can say:

Small $\sigma \rightarrow$ narrow curve $\rightarrow$ high precision
Big $\sigma \rightarrow$ wide curve $\rightarrow$ low precision

There are two important confidence levels that you should be familiar with, the 68% confidence level and the 95% confidence level. What does this now mean? Well, consider an experiment where you record 100 readings. As we have learnt before, the "true" and most likely value for any of these readings will be μ. In reality, our readings will somewhat scatter around that unknown value μ. Some of our readings will be larger than μ and some will be smaller. As you will remember, μ is the maximum of our normal distribution. With the help of the standard deviation, we can now define an interval of $\mu \pm \sigma$ as shown in figure 4. According to the rules of statistics, 68 of our 100 readings will now lie within that interval. We then say that we have an uncertainty at the 68% confidence level. If you were to define the interval a bit wider as $\mu \pm 2\sigma$, then 95 of our 100 readings would fall within that interval. This would be the uncertainty at the 95% confidence level.

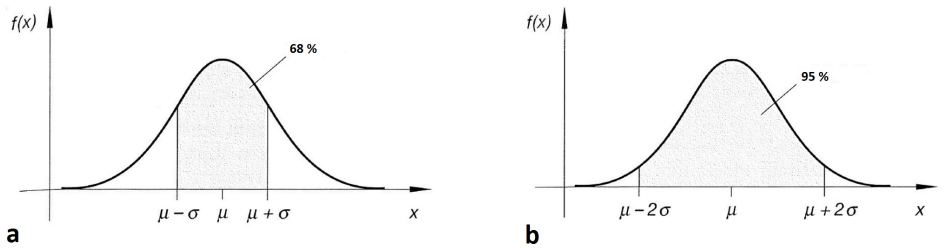

Figure 4: a: Mean with 68% confidence interval. b: Mean with 95% confidence interval.

2 Analysis of a series of measurements

In the previous section, you have learnt what a normal distribution is and what the two parameters mean μ and standard deviation σ mean. But there is one crucial problem, an that is that both parameters are unknown to us. It is now our task to find reasonably well estimates of μ and σ.
Let us start with the mean. The easiest approximation to the true mean μ is the arithmetic mean $\bar{x}$. Please note that the arithmetic mean is not necessarily the true mean and may vary from it, however, it is the easiest way to determine a mean of a set of measurements. We will denote the arithmetic mean as $\bar{x}$ to differentiate it from the "true" but unknown mean μ. From now on, every time we talk about the mean, we talk about the arithmetic mean. The arithmetic mean can simply be calculated by adding up all your readings and then divide by the number of readings you took. So in our case, $x_1, x_2, ..., x_n$ represent our readings and n is the total number of readings. Therefore, the arithmetic mean $\bar{x}$ can be calculated with the following equation.

$$\bar{x} = \frac{1}{n}\left(x_1 + x_2 + ... + x_n\right) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The mean variation of the different readings x_i around the arithmetic mean $\bar{x}$ can be described by the standard deviation σ. The standard deviation σ describes how reliable and precise our set of measurements.

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

This is all good and well, but what we need in the end is the standard deviation of the mean denoted as $\sigma_{\bar{x}}$. It describes the mean variation of the mean that has been calculated from a multitude of single readings. The standard deviation of the mean $\sigma_{\bar{x}}$ describes how precise the mean is and is always smaller than the standard deviation σ.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

As we have learnt so far, the mean values from different measurement series of the same experiment of size n scatter around the true but unknown mean. Therefore we have to state the arithmetic mean $\bar{x}$ with an interval that covers the true mean with a certain probability P. This interval is defined with the following two limits.

$$\begin{aligned} \text{Lower interval limit: } & \bar{x} - \left(\frac{\sigma}{\sqrt{n}}\right) = \bar{x} - \sigma_{\bar{x}} \\ \text{Upper interval limit: } & \bar{x} + \left(\frac{\sigma}{\sqrt{n}}\right) = \bar{x} + \sigma_{\bar{x}} \end{aligned}$$

These interval limits lead to the 68% confidence level which you have gotten to know already in the previous section. If we change the interval limits to

$$\begin{aligned} \text{Lower interval limit: } & \bar{x} - \left(\frac{2\sigma}{\sqrt{n}}\right) = \bar{x} - 2\sigma_{\bar{x}} \\ \text{Upper interval limit: } & \bar{x} + \left(\frac{2\sigma}{\sqrt{n}}\right) = \bar{x} + 2\sigma_{\bar{x}} \end{aligned}$$

then we would be at the 95% confidence level. When performing error calculation, we usually state the interval as

$$x = \bar{x} \pm \frac{\sigma}{\sqrt{n}} = \bar{x} \pm \sigma_{\bar{x}}$$

The half width of that interval, i.e. the distance between the lower or upper limit to the arithmetic mean $\bar{x}$, is called uncertainty and is denoted as Δx. Therefore the result can also be written as

$$x = \bar{x} \pm \Delta x$$

3 Error Propagation

Under error propagation we basically understand a set of simple mathematical rules that allow you to propagate the errors of one or several quantities to another quantity. In order to shed some light on this, let us assume you have measured some quantities x_1, x_2, ..., x_n and also calculated their respective errors or uncertainties Δx_1, Δx_2, ..., Δx_n. It may very well happen that you now want to calculate another quantity z, that depends on one or several of the initial quantities you have measured. The question is now...what is the uncertainty of z? Well, you have to propagate those errors from your initial quantities x_1, x_2, ..., x_n to the new quantity z. If we assume that the errors of your quantities x_1, x_2, ..., x_n are uncorrelated and random, then there are some simple rules that can help you to calculate the error of z.

3.1 Addition Rule

This rule applies if z is calculated by adding or subtracting x_1, x_2, ..., x_n. The error of z is then simply the square root of the sum of squares of the errors of x_1, x_2, ..., x_n, i.e. Δx_1, Δx_2, ..., Δx_n.

$$\Delta z = \sqrt{(\Delta x_1 + \Delta x_2 + ... + \Delta x_n)^2}$$

If

$z = x_1 + x_2$ or
$z = x_1 - x_2$,

the error in z in either case is calculated as

$\Delta z = \sqrt{(\Delta x_1 + \Delta x_2)^2}$

If z is the combination of sums and differences, such as

$z = x_1 + x_2 - x_3 + x_4$ or
$z = x_1 + x_2 - (x_3 + x_4) = x_1 + x_2 - x_3 - x_4$,

the uncertainties would still add in quadrature in order to get Δz:

3.2 Quotient Rule

If z is calculated by multiplying or dividing several quantities x_1, x_2, ..., x_n that have errors Δx_1, Δx_2, ..., Δx_n, then the quotient rule is used to calculate the error in z. The quotient rule states that the fractional uncertainties add in quadrature.

$$\Delta z = |z| \sqrt{\left(\frac{\Delta x_1}{|x_1|}\right)^2 + \left(\frac{\Delta x_2}{|x_2|}\right)^2 + ... + \left(\frac{\Delta x_n}{|x_n|}\right)^2}$$

The same applies to a combination of multiplication and division such as

$z = \frac{x_1 x_2}{x_3 x_4}$,

for which the error is given as

$$\Delta z = |z| \sqrt{\left(\frac{\Delta x_1}{|x_1|}\right)^2 + \left(\frac{\Delta x_2}{|x_2|}\right)^2 + \left(\frac{\Delta x_3}{|x_3|}\right)^2 + \left(\frac{\Delta x_4}{|x_4|}\right)^2}$$

3.3 Power Rule

The power rule applies when we need to raise a quantity x that has a certain error Δx to a power in order to calculate z. In that scenario, in order to find the correct error in z, we need to use the power rule. If z is calculated by raising x to any power n, and n is an exact number, then Δz calculates as

$$\Delta z = |n| x^{n-1} \Delta x$$

If

$z = \frac{1}{x} = x^{-1}$

then the error of z calculates as follows

$\Delta z = |-1| x^{-1-1} \Delta x = \frac{\Delta x}{x^2}$

3.4 Factor Rule

The factor rule is relevant when you have a constant factor in your calculation. It is by far the easiest of all the error propagation rules. All you need to remember here is that when propagating an error of a quantity that has a constant factor, then you will take the absolute value of that constant factor into account. That means negative constants will be positive. So if $z = Cx$, then Δz calculates as

$$\Delta z = |C| \Delta x$$

3.5 Complicated Formulas

The above rules are reasonably simple and easy to apply. Unfortunately, error propagation is not always that easy. You could encounter a much more complex relationship between two or several quantities of which you then need to propagate their errors. The good news is, for most cases that you will encounter during your undergraduate degree, you can propagate the error by combining the above equations. When doing this, you must pay special attention to one crucial consideration, and that is that the errors being propagated must be uncorrelated! This is extremely important in order to get the right solution. This means you sometimes have to rewrite your equations so that the same variable does not appear more than once.

Let's take a look at some examples to illustrate this a little bit further.

First we want to look at what we mean by the errors propagated must be uncorrelated.

Example 1: $z = x^2$ and we want to calculate Δz
If we were to give this exercise during class, a good student will instantly see that we have to use the power rule.

$$\Delta z = |n|x^{n-1}\Delta x = |2|x^{2-1}\Delta x = 2x\Delta x$$

Another student however might say that this is all well and good, but there is also another way to it. This student can argue that x^2 is nothing different than x times x, so basically just a multiplication. Therefore it must also be possible to use the quotient rule.

$$\frac{\Delta z}{|z|} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta x}{x}\right)^2} = \sqrt{2\left(\frac{\Delta x}{x}\right)^2} = \sqrt{2}\left(\frac{\Delta x}{x}\right)$$

And therefore

$$\Delta z = \sqrt{2}|z|\left(\frac{\Delta x}{x}\right) = \sqrt{2}x^2\left(\frac{\Delta x}{x}\right) = \sqrt{2}x\Delta x$$

The problem is, the two solutions are not the same: $2x\Delta x \neq \sqrt{2}x\Delta x$. Now which of our two student is right? The correct way to calculate Δz is with the power rule of course, as that's exactly what this rule was made for. The reason the quotient rule does not give you the correct solution, even though x^2 is indeed nothing different than x times x, is that errors being propagated must not be correlated. In this example, the two variables x and x have correlated errors, simply because any variable is correlated with itself and therefore the quotient rule will lead to a false solution.

But what do we do when we have a more complex equation where we need to use a combination of these error propagation rules but see that one or several quantities appear more than once. Let's take a look at example 2 and go through it step-by-step.

Example 2: $z = \frac{x}{x+y}$ and once again we want to calculate Δz
As you can see, nominator and denominator both contain x, so x appears twice in our equation. In order to solve that problem, we need to do a simple mathematical operation that will change the appearance of our equation slightly. In this case, we simply divide nominator and denominator by x. We are allowed to do that as a multiplication (we are multiplying with x^{-1}) performed on nominator and denominator does not change the equation itself, only its appearance.

$$z = \frac{x}{x+y} = \frac{\frac{x}{x}}{\frac{x+y}{x}} = \frac{1}{\frac{x}{x}+\frac{y}{x}} = \frac{1}{1+\frac{y}{x}}$$

Now our equation contains x once and y once and we do not need to be concerned about propagating correlated errors. But how do we calculate Δz now? In this case you need to apply three of our error propagation rules. First the quotient rule to calculate the error on y/x. After that apply the addition rule to get the error of $1 + y/x$, and finally the power rule to get the error of $(1 + y/x)^{-1}$.

1. Quotient Rule

$$\Delta\left(\frac{x}{y}\right) = \left|\frac{x}{y}\right| \sqrt{\left(\frac{\Delta x}{|x|}\right)^2 + \left(\frac{\Delta y}{|y|}\right)^2}$$

2. Addition Rule

$$\Delta\left(1+\frac{x}{y}\right) = \sqrt{0 + \left|\frac{x}{y}\right| \sqrt{\left(\frac{\Delta x}{|x|}\right)^2 + \left(\frac{\Delta y}{|y|}\right)^2}}$$

3. Power Rule
$z = \frac{1}{1+\frac{x}{y}} = \left(1+\frac{x}{y}\right)^{-1}$, therefore

$$\Delta z = |-1|\left(1+\frac{x}{y}\right)^{-1-1} \Delta\left(1+\frac{x}{y}\right) = \left(1+\frac{x}{y}\right)^{-2} \sqrt{\left|\frac{x}{y}\right| \sqrt{\left(\frac{\Delta x}{|x|}\right)^2 + \left(\frac{\Delta y}{|y|}\right)^2}}$$

$$= \frac{\left(\left|\frac{x}{y}\right| \sqrt{\left(\frac{\Delta x}{|x|}\right)^2 + \left(\frac{\Delta y}{|y|}\right)^2}\right)^{\frac{1}{2}}}{\left(1+\frac{x}{y}\right)^2}$$

This might no look fairly complicated, but when doing this with actual numbers the results after each rule application are much simpler and the entire calculation looks much friendlier. When you are doing an error propagation for your lab reports, we recommend you use MS Excel for it. You only have to programme an Excel sheet once and insert all the equations you need and the programme will then calculate and propagate the errors for all your measurements automatically.

4 Regression Curves

4.1 Introduction

In many technical applications there is often the question whether there is a functional relation between two variables x and y. In this section we want to understand how we can determine a curve from a series of n measurements that describes the relation between two variables in the best possible way. In order to do that we conduct an experiment from which we receive a series of measurement pairs $(x_i; y_i)$ with $i = 1, 2, ..., n$. These data points can then be depicted in an x-y-diagram. We now want to find a curve $y = f(x)$ that best fits those measurement points in our x-y-diagram. If we know the functional definition of such a curve, we can then estimate the dependent value y of any independent variable x.
To deepen our understanding of this, let's look at a specific example and go through it step-by-step. In an experiment we want to investigate the temperature dependence of an electrical resistor over a predefined temperature range. We measured the resistance at eight different temperatures (table 2). We can now put these measured temperature/resistance pairs $(T_i; R_i)$ in a diagram (figure 5).

Table 2: Example measurement protocol

i	1	2	3	4	5	6	7	8
T_i[°C]	20	25	30	40	50	60	65	80
R_i[Ω]	16.30	16.44	16.61	16.81	17.10	17.37	17.38	17.86

By looking at the data points it is now possible to draw a linear curve that will fairly well match our data points. The scatter around the linear curve can be explained as random measurement errors which we simply can accept. We only took one reading at every temperature However, would we have taken for example 10 readings at every temperature, then all 10 readings would probably be slightly different to each other due to random measurement errors. In figure 5 the linear curve was adjusted so that the data points are fairly well balanced out to either side of the curve. However, the selection of such a curve is very subjective and different investigators will inevitably choose different regression curves. That is why we need some mathematical bits that tell us exactly which curve represents our data cloud in the best way. This mathematical technique has been given to us by the great mathematician Carl Friedrich Gauß and is called "Least Squares Method".

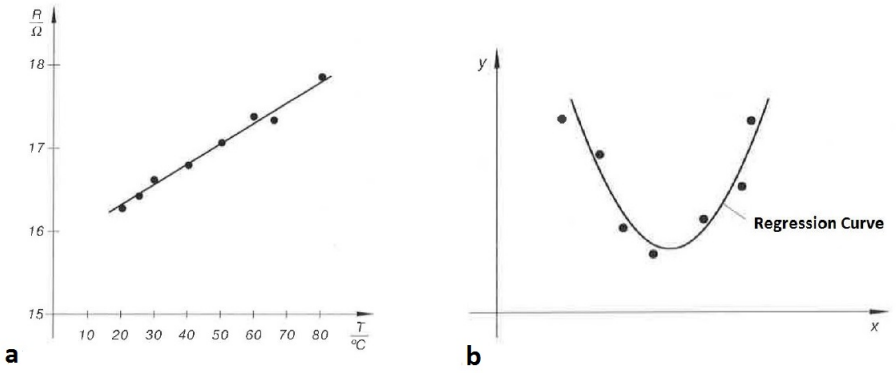

Figure 5: a: Temperature dependence of an electrical resistor. The data points scatter around a straight line. b: Second order regression curve (i.e. parabola.)

4.2 Least Squares Method

Before we really start with the description of the least squares method, we want to make the following assumption. The variable x, which is our independent variable, does have some random measurement errors. However, we will treat it as though these errors are negligible and won't take them into account. So we assume that the x values will stay the same even if we were to measure them over and over again. The dependent variable y (i.e. dependent of x) on the other hand is a random variable because these values will vary slightly with every repeat of the experiment even though we are using exactly the same x. So the y values will scatter around a middle value. As a measure of the difference between measurement point and regression curve, we introduce the vertical distance v_i, i.e. the distance in ordinate direction (figure 6). So we can say that the distance from the data point $P_i = (x_i; y_i)$ to the yet unknown regression curve $y = f(x)$ is

$$v_i = y_i - f(x_i)$$

With that in mind, Gauß's idea was now that the best fit for our cloud of data points P_i, would be a curve for which the sum of the differences squared of all data points would give a minimum. We square the differences because a data point can lie below or above the best fit curve and hence the difference could be negative. We don't have to worry about that if we simply square the differences before adding them up.

$$S = \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} (y_i - f(x_i))^2 \rightarrow Minimum$$

But to what type of function do I try to fit my data? The answer to this

is...it depends! It depends on whether you might have prior knowledge and expect a certain kind of behaviour. Or by simply looking at the distribution of your data points you might have a functional relation in mind. If your data cloud looks somewhat similar to that in figure 5 where the data points lie more or less in a straight line, then it is obvious that it would be best to choose a linear regression curve

$$y = ax + b$$

But in other cases it might be better to choose a quadratic curve (i.e. parabola) as shown in figure 5.

$$y = ax^2 + bx + c$$

In either case, the curve parameters (a and b or a, b and c) can be determined so that the curve fits our data points in an optimal way. Once we have decided which function type (linear, quadratic, exponential, logarithmic etc.) we want to use for our data cloud, the function parameters can be determined so that the sum of the squares of these parameters has a minimum.

$$S(a; b; ...) = \sum_{i=1}^{n} (Y_i - f(x_i))^2$$

Following the rules of differential calculus, this is the case when the first order partial derivatives of $S(a; b; ...)$ disappear (i.e. are zero).

$$\frac{\partial S}{\partial a} = 0$$
$$\frac{\partial S}{\partial b} = 0$$

This then leads to a system of equations from which the parameters can be determined. The above equations are usually non-linear and therefore can only be solved numerically with long computing times. Only in very simple cases will it result in a linear system of equations that can be solved by hand with the Gaussian algorithm.

We decided for this handout not to include an example of how to determine the parameters a and b of a linear regression curve by hand calculations. The reason is that living in the 21^{st} century, we now have computers and smart software tools such as MS Excel and Matlab to just mention a couple that do all the hard work for us. Nevertheless it is important to have a rough idea of what these programmes do when you feed your data into them.

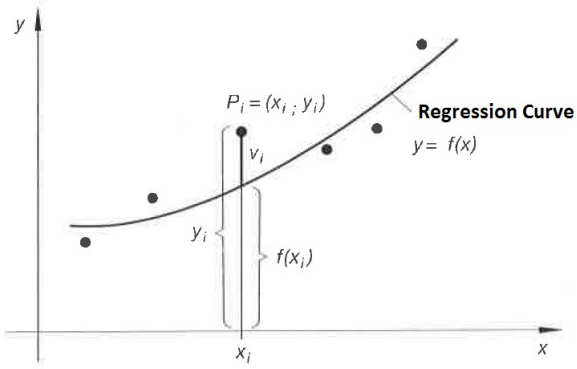

Figure 6: Datapoints $P_i = (x_i; y_i)$ with regression curve.

4.3 Non-linear Problems

Linear solutions, that means fitting a linear curve to our data points is always much easier than other higher degree or exponential functions. However, many non-linear problems can be treated as if they were linear by performing a transformation. This is a useful trick that is often applied in many science and engineering disciplines.
Let's say we have a exponential function of the form

$$y = ae^{bx}$$

After using the ln function we get

$$lny = ln(ae^{bx}) = lna + ln(e^{bx}) = lna + x = bx + lna$$

Now we use the following transformation:

$$u = x$$

$$v = lny$$

$$c = b$$

$$d = lna$$

which leads to the following linear form

$$v = cu + d$$

This means if we plot lny on the vertical axis (i.e. ordinate) and x (like before) on the horizontal axis (i.e. abscissa) we receive a straight line (figure 7). The ordinate axis will therefore be a logarithmic axis while the abscissa stays linear. During the transformation $(x_i; y_i)$ go over into $(u_i; v_i) = (x_i; lny)$ with which we can perform a linear regression. This will then give us the

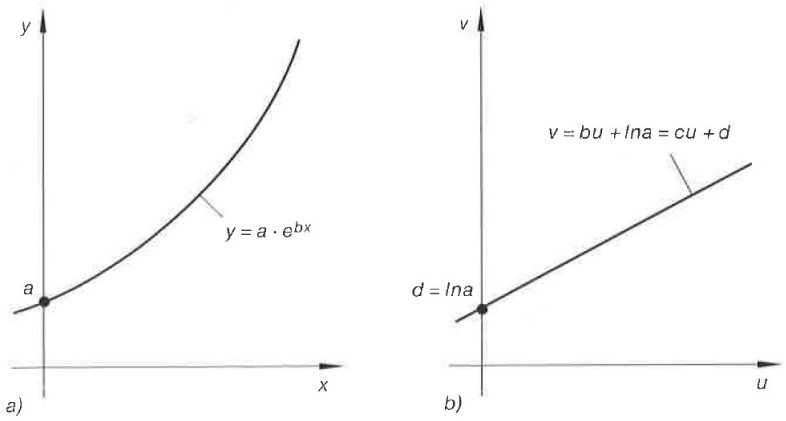

Figure 7: Transformation of a exponential function into a linear function to simply the regression calculation.

two parameters c and d which we can transform back and use to calculate the initial parameters a and b.

$$lna = d \rightarrow a = e^d$$

$$b = c$$

5 Further Reading

Hughes, I. and Hase, T., 2010. *Measurements and their Uncertainties: A Practical Guide to Modern Error Analysis.* Oxford: OUP Oxford.

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