The inverse of a matrix

Introduction

In this leaflet we explain what is meant by an inverse matrix and how it is calculated.

1. The inverse of a matrix

The inverse of a square \( n \times n \) matrix \( A \), is another \( n \times n \) matrix denoted by \( A^{-1} \) such that

\[
AA^{-1} = A^{-1}A = I
\]

where \( I \) is the \( n \times n \) identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be singular. Only non-singular matrices have inverses.

2. A formula for finding the inverse

Given any non-singular matrix \( A \), its inverse can be found from the formula

\[
A^{-1} = \frac{\text{adj} \ A}{|A|}
\]

where \( \text{adj} \ A \) is the adjoint matrix and \( |A| \) is the determinant of \( A \). The procedure for finding the adjoint matrix is given below.

3. Finding the adjoint matrix

The adjoint of a matrix \( A \) is found in stages:

(1) Find the transpose of \( A \), which is denoted by \( A^T \). The transpose is found by interchanging the rows and columns of \( A \). So, for example, the first column of \( A \) is the first row of the transposed matrix; the second column of \( A \) is the second row of the transposed matrix, and so on.

(2) The minor of any element is found by covering up the elements in its row and column and finding the determinant of the remaining matrix. By replacing each element of \( A^T \) by its minor, we can write down a matrix of minors of \( A^T \).

(3) The cofactor of any element is found by taking its minor and imposing a place sign according to the following rule

\[
\begin{pmatrix}
+ & - & + & \ldots \\
- & + & - & \ldots \\
+ & - & + & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]
This means, for example, that to find the cofactor of an element in the first row, second column, the sign of the minor is changed. On the other hand to find the cofactor of an element in the second row, second column, the sign of the minor is unaltered. This is equivalent to multiplying the minor by ‘+1’ or ‘−1’ depending upon its position. In this way we can form a matrix of cofactors of $A^T$. This matrix is called the adjoint of $A$, denoted adj $A$.

The matrix of cofactors of the transpose of $A$, is called the adjoint matrix, adj $A$.

This procedure may seem rather cumbersome, so it is illustrated now by means of an example.

Example

Find the adjoint, and hence the inverse, of $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$.

Solution

Follow the stages outlined above. First find the transpose of $A$ by taking the first column of $A$ to be the first row of $A^T$, and so on:

$$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

Now find the minor of each element in $A^T$. The minor of the element ‘1’ in the first row, first column, is obtained by covering up the elements in its row and column to give $\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$ and finding the determinant of this, which is $−7$. The minor of the element ‘3’ in the second column of the first row is found by covering up elements in its row and column to give $\begin{pmatrix} -2 & 2 \\ 0 & 3 \end{pmatrix}$ which has determinant $−6$. We continue in this fashion and form a new matrix by replacing every element of $A^T$ by its minor. Check for yourself that this process gives

$$\text{matrix of minors of } A^T = \begin{pmatrix} -7 & -6 & -10 \\ 14 & 3 & 5 \\ 7 & 0 & 7 \end{pmatrix}$$

Then impose the place sign. This results in the matrix of cofactors, that is, the adjoint of $A$.

$$\text{adj } A = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Notice that to complete this last stage, each element in the matrix of minors has been multiplied by 1 or $−1$ according to its position.

It is a straightforward matter to show that the determinant of $A$ is $21$. Finally

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{21} \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Exercise

1. Show that the inverse of $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ -1 & 3 & 0 \end{pmatrix}$ is $\frac{1}{4} \begin{pmatrix} -3 & 6 & -7 \\ -1 & 2 & -1 \\ 5 & -6 & 5 \end{pmatrix}$. 

5.5.2  

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**Adjoint of Matrix:**

Adjoint or Adjugate Matrix of a square matrix is the transpose of the matrix formed by the cofactors of elements of determinant |A|.

To calculate adjoint of matrix we have to follow the procedure:

a) Calculate Minor for each element of the matrix.
b) Form Cofactor matrix from the minors calculated.
c) Form Adjoint from cofactor matrix.

For an example we will use a matrix A:

Matrix A =

<table>
<thead>
<tr>
<th>a11</th>
<th>a12</th>
<th>a13</th>
</tr>
</thead>
<tbody>
<tr>
<td>a21</td>
<td>a22</td>
<td>a23</td>
</tr>
<tr>
<td>a31</td>
<td>a32</td>
<td>a33</td>
</tr>
</tbody>
</table>

**Step 1: Calculate Minor for each element.**

To calculate the minor for an element we have to use the elements that do not fall in the same row and column of the minor element.

Minor of a11 = $M_{11} = \begin{vmatrix} a12 & a13 \\ a21 & a23 \\ a31 & a33 \end{vmatrix} = a22 \cdot a33 - a32 \cdot a23$

Minor of a12 = $M_{12} = \begin{vmatrix} a11 & a13 \\ a21 & a23 \\ a31 & a33 \end{vmatrix} = a21 \cdot a33 - a31 \cdot a23$

Minor of a13 = $M_{13} = \begin{vmatrix} a11 & a12 \\ a21 & a22 \\ a31 & a32 \end{vmatrix} = a21 \cdot a32 - a31 \cdot a22$

Minor of a21 = $M_{21} = \begin{vmatrix} a11 & a13 \\ a22 & a23 \\ a31 & a33 \end{vmatrix} = a12 \cdot a33 - a32 \cdot a13$

Similarly

$M_{22} = a11a33 - a31a13$

$M_{31} = a12a23 - a22a13$

$M_{33} = a11a22 - a21a12$

$M_{23} = a11a32 - a31a12$

$M_{32} = a12a23 - a22a13$
Step 2: Form a matrix with the minors calculated.

\[
\text{Matrix of Minors} = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\]

Step 3: Finding the cofactor from Minors:

**Cofactor:** A signed minor is called cofactor.

The cofactor of the element in the \(i^{th}\) row, \(j^{th}\) column is denoted by \(C_{ij}\)

\[C_{ij} = (-1)^{i+j} M_{ij}\]

\[
\text{Matrix of Cofactors} = \begin{bmatrix}
(-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & (-1)^{1+3}M_{13} \\
(-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & (-1)^{2+3}M_{23} \\
(-1)^{3+1}M_{31} & (-1)^{3+2}M_{32} & (-1)^{3+3}M_{33}
\end{bmatrix}
\]

So,

\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix} = \begin{bmatrix}
M_{11} & -M_{12} & M_{13} \\
-M_{21} & M_{22} & -M_{23} \\
M_{31} & -M_{32} & M_{33}
\end{bmatrix}
\]

Step 4: Calculate adjoint of matrix:

To calculate adjoint of matrix, just put the elements in rows to columns in the cofactor matrix. i.e convert the elements in first row to first column, second row to second column, third row to third column.

\[
\text{Adjoint of Matrix} = \begin{bmatrix}
C_{11} & C_{21} & C_{31} \\
C_{12} & C_{22} & C_{32} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}
\]
Adjugate matrix

From Wikipedia, the free encyclopedia

In linear algebra, the adjugate or classical adjoint (occasionally referred to as adjunct) of a square matrix is the transpose of the cofactor matrix.

The adjugate has sometimes been called the "adjoint", but today the "adjoint" of a matrix normally refers to its corresponding adjoint operator, which is its conjugate transpose.

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Definition

The adjugate of $A$ is the transpose of the cofactor matrix $C$ of $A$:

$$\text{adj}(A) = C^T.$$  

In more detail: suppose $R$ is a commutative ring and $A$ is an $n \times n$ matrix with entries from $R$.

- The $(ij)$ minor of $A$, denoted $M_{ij}$, is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting row $i$ and column $j$ of $A$.

- The cofactor matrix of $A$ is the $n \times n$ matrix $C$ whose $(ij)$ entry is the $(ij)$ cofactor of $A$:

$$C_{ij} = (-1)^{i+j} M_{ij}.$$  

- The adjugate of $A$ is the transpose of $C$, that is, the $n \times n$ matrix whose $(ij)$ entry is the $(ji)$ cofactor of $A$:

$$\text{adj}(A)_{ij} = C_{ji}.$$  

The adjugate is defined as it is so that the product of $A$ and its adjugate yields a diagonal matrix whose diagonal entries are $\det(A)$:

$$A \text{adj}(A) = \det(A) I.$$  

$A$ is invertible if and only if $\det(A)$ is an invertible element of $R$, and in that case the equation above yields:

$$\text{adj}(A) = \det(A) A^{-1},$$

$$A^{-1} = \det(A)^{-1} \text{adj}(A).$$
Examples

2 × 2 generic matrix

The adjugate of the 2 × 2 matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

is

\[
\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

It is seen that \( \text{det}(\text{adj}(A)) = \text{det}(A) \) and \( \text{adj}(\text{adj}(A)) = A \).

3 × 3 generic matrix

Consider the 3 × 3 matrix

\[
A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\]

Its adjugate is the transpose of the cofactor matrix

\[
C = \begin{pmatrix} + |A_{22} A_{33}| - |A_{21} A_{33}| + |A_{21} A_{32}| \\ - |A_{12} A_{33}| + |A_{11} A_{33}| - |A_{11} A_{32}| \\ + |A_{12} A_{32}| - |A_{11} A_{32}| + |A_{11} A_{22}| \end{pmatrix} = \begin{pmatrix} + |5 6| - |4 6| + |4 5| \\ - |8 9| + |7 9| - |7 8| \\ + |2 3| - |1 3| + |1 2| \end{pmatrix}
\]

So that we have

\[
\text{adj}(A) = \begin{pmatrix} + |A_{22} A_{33}| - |A_{21} A_{33}| + |A_{21} A_{32}| \\ - |A_{12} A_{33}| + |A_{11} A_{33}| - |A_{11} A_{32}| \\ + |A_{12} A_{32}| - |A_{11} A_{32}| + |A_{11} A_{22}| \end{pmatrix} = \begin{pmatrix} + |5 6| - |2 3| + |2 3| \\ - |8 9| + |8 9| - |5 6| \\ + |4 6| - |1 3| + |1 3| \end{pmatrix}
\]

where

\[
\begin{vmatrix} A_{im} & A_{in} \\ A_{jm} & A_{jn} \end{vmatrix} = \text{det} \left( \begin{pmatrix} A_{im} & A_{in} \\ A_{jm} & A_{jn} \end{pmatrix} \right)
\]

Therefore \( C \), the adjugate of \( A \), is
\[
C = \begin{pmatrix}
-3 & 6 & -3 \\
6 & -12 & 6 \\
-3 & 6 & -3
\end{pmatrix}
\]

Note that the adjugate is the transpose of the cofactor matrix. Thus, for instance, the (3,2) entry of the adjugate is the (2,3) cofactor of A.

**3 × 3 numeric matrix**

As a specific example, we have

\[
\text{adj} \begin{pmatrix}
-3 & 2 & -5 \\
-1 & 0 & -2 \\
3 & -4 & 1
\end{pmatrix} = \begin{pmatrix}
-8 & 18 & -4 \\
-5 & 12 & -1 \\
4 & -6 & 2
\end{pmatrix}
\]

The -6 in the third row, seconds column of the adjugate was computed as follows:

\[
(-1)^{2+3} \det \begin{pmatrix}
-3 & 2 \\
3 & -4
\end{pmatrix} = -((-3)(-4) - (3)(2)) = -6.
\]

Again, the (3,2) entry of the adjugate is the (2,3) cofactor of A. Thus, the submatrix

\[
\begin{pmatrix}
-3 & 2 \\
3 & -4
\end{pmatrix}
\]

was obtained by deleting the second row and third column of the original matrix A.

**Properties**

The adjugate has the properties

\[
\begin{align*}
\text{adj}(I) &= I, \\
\text{adj}(AB) &= \text{adj}(B) \text{adj}(A), \\
\text{adj}(cA) &= c^{n-1} \text{adj}(A)
\end{align*}
\]

for \(n \times n\) matrices A and B. The second line follows from equations \(\text{adj}(B)\text{adj}(A) = \det(B)B^{-1} \det(A)A^{-1} = \det(AB)(AB)^{-1}\).

Substituting in the second line \(B = A^{m-1}\) and performing the recursion, one gets for all integer \(m\)

\[
\text{adj}(A^m) = \text{adj}(A)^m.
\]

The adjugate preserves transposition:

\[
\text{adj}(A^T) = \text{adj}(A)^T.
\]

Furthermore,

\[
\begin{align*}
\det(\text{adj}(A)) &= \det(A)^{n-1}, \\
\text{adj}(\text{adj}(A)) &= \det(A)^{n-2}A
\end{align*}
\]

and, if \(\det(A)\) is a unit, then \(\det(\text{adj}(A)) = \det(A)\) and \(\text{adj}(\text{adj}(A)) = A\).

**Inverses**

As a consequence of Laplace's formula for the determinant of an \(n \times n\) matrix A, we have

\[
A \text{adj}(A) = \text{adj}(A) A = \det(A) I_n \quad (\ast)
\]

where \(I_n\) is the \(n \times n\) identity matrix. Indeed, the \((i,i)\) entry of the product \(A \text{adj}(A)\) is the scalar product of row \(i\) of \(A\) with row \(i\) of the cofactor matrix \(C\), which is simply the Laplace formula for \(\det(A)\) expanded by row \(i\). Moreover, for \(i \neq j\) the \((ij)\) entry of the product
is the scalar product of row $i$ of $A$ with row $j$ of $C$, which is the Laplace formula for the determinant of a matrix whose $i$ and $j$ rows are equal and is therefore zero.

From this formula follows one of the most important results in matrix algebra: A matrix $A$ over a commutative ring $R$ is invertible if and only if $\det(A)$ is invertible in $R$.

For if $A$ is an invertible matrix then

$$1 = \det(I_n) = \det(\mathbf{AA}^{-1}) = \det(A) \det(A^{-1}),$$

and equation (*) above shows that

$$A^{-1} = \det(A)^{-1} \text{adj}(A).$$

See also Cramer's rule.

**Characteristic polynomial**

If $p(t) = \det(A - t \mathbf{I})$ is the characteristic polynomial of $A$ and we define the polynomial $q(t) = (p(0) - p(t))/t$, then

$$\text{adj}(A) = q(A) = -(p_1 \mathbf{I} + p_2 A + p_3 A^2 + \cdots + p_n A^{n-1}),$$

where $p_j$ are the coefficients of $p(t)$,

$$p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n.$$  

**Jacobi's formula**

The adjugate also appears in Jacobi's formula for the derivative of the determinant:

$$\frac{d}{d\alpha} \det(A) = \text{tr} \left( \text{adj}(A) \frac{dA}{d\alpha} \right).$$

See also

- Trace diagram

**References**


**External links**

- Online matrix calculator (determinant, track, inverse, adjoint, transpose) (http://www.stud.fecv.vutbr.cz/~xvapen02/vypocty/matrireg.php?language=english) Compute Adjugate matrix up to order 8
- "adjugate of { a, b, c }, { d, e, f }, { g, h, i }" (http://www.wolframalpha.com/input/?i=adjugate+of+{+a%2C+b%2C+c%2C}+%2C+{+d%2C+e%2C+f%2C}+%2C+{+g%2C+h%2C+i%2C}+}. Wolfram Alpha. http://www.wolframalpha.com/input/?i=adjugate+of+{+a%2C+b%2C+c%2C}+%2C+{+d%2C+e%2C+f%2C}+%2C+{+g%2C+h%2C+i%2C}+). Retrieved from "http://en.wikipedia.org/w/index.php?title=Adjugate_matrix&oldid=536232230"

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